Abstract

Let $G$ be a graph. For two vertices $u$ and $v$ in $G$, we denote $d(u, v)$ the distance between $u$ and $v$. Let $j, k$ be positive integers with $j \geq k$. An $L(j, k)$-labelling for $G$ is a function $f : V(G) \to \{0, 1, 2, \ldots\}$ such that for any two vertices $u$ and $v$, $|f(u) - f(v)|$ is at least $j$ if $d(u, v) = 1$; and is at least $k$ if $d(u, v) = 2$. The span of $f$ is the difference between the largest and the smallest numbers in $f(V)$. The $\lambda_{j,k}$-number for $G$, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j, k)$-labellings of $G$. We introduce a new parameter for a tree $T$, namely, the maximum ordering-degree, denoted by $M(T)$. Combining this new parameter and the special family of infinite trees introduced by Chang and Lu [3], we present upper and lower bounds for $\lambda_{j,k}(T)$ in terms of $j, k, M(T)$, and $\Delta(T)$ (the maximum degree of $T$). For a special case when $j \geq \Delta(T)k$, the upper and the lower bounds are $k$ apart. Moreover, we completely determine $\lambda_{j,k}(T)$ for trees $T$ with $j \geq M(T)k$.

1 Introduction

Motivated by the channel assignment problem (cf. Hale [14]), distance-two labelling was introduced and formulated by Griggs and Yeh [13]. Suppose a number of transmitters or stations are given. We ought to assign to each transmitter with a channel, which is a non-negative integer, such that the interference is avoided. In order to reduce the interference, any pair of ‘close’ transmitters must receive different channels,
and any pair of ‘very close’ transmitters (between which stronger interference might occur) must receive channels that are at least two apart. The objective is to find a valid assignment such that the span of the channels used is minimized.

Graphs are natural models for the above mentioned channel assignment problem. We represent each transmitter by a vertex, where each pair of very close transmitters are connected by an edge between the corresponding vertices. A pair of close transmitters are represented by vertices of distance two apart in the graph. Hence, an \(L(2,1)\)-labelling is defined as follows. For a given graph \(G\), an \(L(2,1)\)-labelling of \(G\) is a function \(f : V(G) \rightarrow \{0, 1, 2, 3, \ldots\}\) such that the following hold, where \(d(u,v)\) is the distance between \(u\) and \(v\) in \(G\):

\[
|f(u) - f(v)| \geq \begin{cases} 
2, & \text{if } d(u,v) = 1; \\
1, & \text{if } d(u,v) = 2.
\end{cases}
\]

The span of \(f\), denoted by \(\text{sp}(f)\), is the difference of the largest and the smallest labels assigned to vertices, that is, \(\text{sp}(f) = \max\{f(V)\} - \min\{f(V)\}\). The \(\lambda\)-number of \(G\), denoted by \(\lambda(G)\), is the minimum span over all \(L(2,1)\)-labellings for \(G\). Since Griggs and Yeh’s first paper [13], \(L(2,1)\)-labelling has been studied extensively (cf. most of the References).

For a more general setting, Griggs and Yeh [13] proposed the study of a labelling \(f\) such that \(|f(u) - f(v)| \geq m_i\) if \(d(u,v) = i\) for \(1 \leq i \leq N\), where \(N\) is a positive integer and \(m_1 \geq m_2 \geq \ldots \geq m_N \geq 0\) are given numbers. If \(N = m_1 = 1\), \(f\) is the same as an ordinary vertex-coloring. If \(N = 2\) and \(m_1 \geq m_2 \geq 1\) are integers, then \(f\) is called an \(L(m_1,m_2)\)-labelling. That is, for given integers \(j \geq k \geq 1\), the \(L(j,k)\)-labelling of \(G\) is a function \(f\) on \(V(G)\) such that the following hold:

\[
|f(u) - f(v)| \geq \begin{cases} 
\bar{j}, & \text{if } d(u,v) = 1; \\
k, & \text{if } d(u,v) = 2.
\end{cases}
\]

The span of \(f\) is defined the same as an \(L(2,1)\)-labelling. The \(\lambda_{j,k}\)-number of \(G\), denoted by \(\lambda_{j,k}(G)\), is the minimum span over all \(L(j,k)\)-labellings for \(G\). An \(L(j,k)\)-labelling \(f\) is called optimal if \(\text{sp}(f) = \lambda_{j,k}(G)\), and in this case \(f\) is also called a \(\lambda_{j,k}\)-labelling. For the case \(k = 1\), \(\lambda_{j,1}(G)\) is denoted by \(\lambda_j(G)\).

For a graph \(G\), let \(\Delta(G)\) denote the maximum degree of \(G\). Georges and Mauro [8] studied \(L(j,k)\)-labellings for general values of \(j\) and \(k\). Among other results shown in [8], there are bounds of \(\lambda_{j,k}(G)\) for all graphs \(G\), in terms of \(i\), \(j\) and
\(\Delta(G)\), complete solutions (for all \(j \geq k \geq 1\)) of \(\lambda_{j,k}(G)\) for special families of graphs including paths and cycles, and partial solutions on the products of paths. Molloy and Salavatipour [17] gave an upper bound of \(\lambda_{j,k}(G)\) for planar graphs \(G\). The \(\lambda_j\)-number was investigated by Chang et al. [5]. For more results on \(L(j,k)\)-labelling of graphs, the reader is referred to the survey articles [1, 19].

In [9], Georges and Mauro derived the \(\lambda_{j,k}\)-number for the infinite regular tree. In addition, for \(x = j/k\), the authors introduced a rational variation \(\lambda_x(G)\) of \(\lambda_{j,k}(G)\), and proved that \(\lambda_x(G)\) is a continuous function, for a fixed graph \(G\). Recently, an even more general distance two labelling using real numbers as labels was introduced and studied by Griggs and Jin [11, 12].

For a graph \(G\) with maximum degree \(\Delta(G)\), it is clear that \(\lambda_{j,k}(G) \geq j + (\Delta(G) - 1)k\). The graphs achieving this bound are called \(\lambda_{j,k}\)-minimal. Chang and Lu [3] studied the structure of \(\lambda_{j,k}\)-minimal graphs. Using a special family of infinite trees, the authors characterized the \(\lambda_{j,k}\)-minimal trees [3].

The aim of this article is to investigate the \(\lambda_{j,k}\)-number for trees \(T\) in general. In Section 2, we introduce a new parameter for trees called the maximum ordering-degree, denoted as \(M(T)\). Combining this new parameter with the special family of infinite trees introduced in [3], in Section 2, we establish a general upper bound of \(\lambda_{j,k}(T)\) for trees, in terms of \(i, j,\) and \(M(T)\). In Section 3 we also use \(M(T)\) to prove a key lemma which is utilized, in Section 4, to show a lower bound of \(\lambda_{j,k}(T)\) in terms of \(M(T)\), provided \(j \geq \Delta(T)k\). Moreover, we give complete solutions of \(\lambda_{j,k}(T)\) for trees with \(j \geq M(T)k\).

We make a note here about the complexity problem of determining the \(\lambda_{j,k}\)-number for trees. Chang and Kuo [4] proved that for a tree \(T\) with maximum degree \(\Delta(T)\), \(\lambda(T) \in \{\Delta(T) + 1, \Delta(T) + 2\}\), and there is a polynomial-time algorithm to determine the exact value of \(\lambda(T)\). Later on, an extended and similar result for \(\lambda_{j,1}(T)\), \(j \geq 1\), was shown by Chang et al. [4]. The problem, however, becomes more complicated for \(k > 1\). Fiala et al. [6] showed that determining the value of \(\lambda_{j,k}(T)\) is NP-complete unless \(j\) is a multiple of \(k\), in which it can be solved polynomially.

Throughout the article, we denote \(T\) as a tree with maximum degree \(\Delta\), unless otherwise indicated.
2 Upper Bound for Trees

We introduce the maximum ordering-degree for trees and the infinite tree used by Chang and Lu [3]. Combining these two notions, we then derive an upper bound of the $\lambda_{j,k}$-number for all trees (Theorem 7).

For the special case $k = 1$, the known upper and lower bounds for the $\lambda_j$-number of trees are due to Griggs and Yeh [13] for $j = 2$, and due to Chang et al. [5] for $j \geq 2$.

**Theorem 1** [5] Let $T$ be a tree with maximum degree $\Delta$. Then $\Delta + j - 1 \leq \lambda_j(T) \leq \min\{2j + \Delta - 2, 2\Delta + j - 2\}$. Moreover, both the lower and the upper bounds are attainable.

For the general case $1 \leq k \leq j$, the following property, observed in [3], follows naturally from the definition.

**Proposition 2** [3] For any positive integers $j \geq k$ and any graph $G$ of maximum degree $\Delta$, we have $\lambda_{j,k}(G) \geq j + (\Delta - 1)k$. Moreover, if the equality holds and $j > k$, then for any $\lambda_{j,k}(G)$-labelling of $G$, each vertex with degree $\Delta$ must be labelled by 0 or $j + (\Delta - 1)k$.

A $\lambda_{j,k}$-minimal tree is a tree $T$ with $\lambda_{j,k}(T) = j + (\Delta - 1)k$. The infinite tree introduced in [3] is defined as follows. For any positive integer $M$, an $[M]$-sequence is a sequence $(b_0, b_1, \ldots, b_t)$ for some $t \geq 0$, so that all the following hold:

(S1) $b_0 = 0$.

(S2) $0 \leq b_i \leq M - 1$ for all $i = 1, 2, \ldots, t$.

(S3) $b_i \geq b_{i-1}$ and $b_i \geq b_{i+1}$ for all odd $i \in \{1, 2, \ldots, t\}$.

(S4) $b_i \neq b_{i+2}$ for all $i = 0, 1, \ldots, t - 2$.

For a positive integer $M$, the infinite tree $T_M$ has the vertex set of all $[M]$-sequences where two vertices $(b_0, b_1, \ldots, b_t)$ and $(c_0, c_1, \ldots, c_{t'})$ are adjacent if $|t - t'| = 1$ and $b_i = c_i$ for $0 \leq i \leq \min\{t, t'\}$.

Below is a characterization of the $\lambda_{j,k}$-minimal trees for large values of $j$. 
**Theorem 3** [3] Let $T$ be a tree of maximum degree $\Delta$. If $j, k$ are integers with $j \geq \Delta k$, then $T$ is $\lambda_{j,k}$-minimal if and only if $T$ is a subtree of $T_{\Delta}$.

The following result was proved implicitly in [3].

**Theorem 4** [3] For any positive integer $M$ and $j \geq k$, $\Delta(T_M) = M$ and $\lambda_{j,k}(T_M) = j + (M - 1)k$. Consequently, $T_M$ is $\lambda_{j,k}$-minimal for any $j \geq k$.

In the following, we introduce the new parameter maximum ordering-degree of a tree $T$. Let $v$ be a vertex in $T$ of degree $n$. The *neighborhood* of $v$, denoted by $N(v)$, is the set of all vertices adjacent to $v$. The set $N(v) \cup \{v\}$ is denoted by $N[v]$. The *neighborhood degree sequence* of $v$, called ND-sequence for brevity, is a non-increasing degree sequence $(h_0(v), h_1(v), \ldots, h_{n-1}(v))$ of $N(v)$. That is, we line up the vertices of $N(v)$ by $(v_0, v_1, v_2, \ldots, v_{n-1})$ where $d(v_0) \geq d(v_1) \geq \cdots \geq d(v_{n-1})$, and record their degrees into the ND-sequence with $d(v_i) = h_i(v)$. We will simply use $(h_0, h_1, \ldots, h_{n-1})$ when $v$ is understood in the context. The maximum ordering-degree of $v$, denoted by $m(v)$, is defined as

$$m(v) = \max\{h_i + i \mid 0 \leq i \leq d(v) - 1\}.$$

The maximum ordering-degree of $T$, denoted by $\mathcal{M}(T)$, is defined as

$$\mathcal{M}(T) = \max\{m(v) \mid v \in V(T)\}.$$

For instance, the maximum ordering-degree for $K_{1, \Delta}$ (a star with $\Delta$ leaves) is $\Delta$.

Throughout the article, we shall simply denote $\mathcal{M}(T)$ by $\mathcal{M}$, when $T$ is clear in the context.

**Lemma 5** Let $T$ be a tree and $v$ a vertex in $T$ with degree $n$. Let $(d_0, d_1, d_2, \ldots, d_{n-1})$ be any ordering of the degrees of $N(v)$ and let $(h_0, h_1, \ldots, h_{n-1})$ be the ND-sequence of $v$. Then $\max\{d_i + i \mid 0 \leq i \leq n - 1\} \geq m(v)$.

**Proof.** Let $m(v) = h_j + j$ for some $0 \leq j \leq n - 1$. Since $\{d_0, d_1, d_2, \ldots, d_{n-1}\} = \{h_0, h_1, \ldots, h_{n-1}\}$ and $h_0 \geq h_1 \geq h_2 \geq \cdots \geq h_{n-1}$, there exists some $t \geq j$ such that $h_j \leq d_t$. Hence, $\max\{d_i + i \mid 0 \leq i \leq n - 1\} \geq d_t + t \geq h_j + j = m(v)$. \qed

The next result emerges directly from the definition of $\mathcal{M}(T)$. 

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Lemma 6 Let $T$ be a tree with maximum degree $\Delta$ and maximum ordering-degree $\mathcal{M}$. Then $\Delta \leq \mathcal{M} \leq 2\Delta - 1$.

For two graphs $G$ and $H$, a graph homomorphism from $G$ to $H$ is an edge-preserving function from $V(G)$ to $V(H)$. If there exists an injective homomorphism from $G$ to $H$, then by composition of functions, one gets $\lambda_{j,k}(G) \leq \lambda_{j,k}(H)$.

We now show a general upper bound of the $\lambda_{j,k}$-number for trees.

Theorem 7 Let $T$ be a tree with maximum ordering-degree $\mathcal{M}$. Let $j, k$ be integers with $j \geq k$. Then $\lambda_{j,k}(T) \leq j + \mathcal{M}k$.

Proof. By Theorem 4, it suffices to find an injective homomorphism from $V(T)$ to $V(T_{\mathcal{M}+1})$. Let $v$ be a vertex with $m(v) = \mathcal{M}$. Suppose $v$ has degree $n$. Let $N(v) = \{v_0, v_1, \cdots, v_{n-1}\}$ with ND-sequence $(h_0, h_1, \cdots, h_{n-1})$, where $h_i$ is the degree of $v_i$. We define a homomorphism by the following recursive process which labels each vertex in $V(T)$ with an $[\mathcal{M} + 1]$-sequence.

Initially, all the vertices are unlabelled. Label $v$ by $(0)$ and its neighbors $v_i$ by $(0, \mathcal{M} - i)$, $0 \leq i \leq n - 1$. Suppose a vertex $u$ has been labelled by $(b_0, b_1, \cdots, b_t)$ for some $t \geq 1$. We next label all the unlabelled neighbors of $u$ (if there exists any). Note, by our labelling scheme, there is only one neighbor of $u$ that has been labelled.

Let $\{u_i | 0 \leq i \leq d(u) - 2\}$ be the set of unlabelled neighbors of $u$, where $d(u_i) = d_i$ and $d_i \geq d_{i+1}$. Observe, $d_i \leq h_i(u)$. We label $u_i$ by $(b_0, b_1, \cdots, b_t, b_{t+1}^{(i)})$ according to the following:

If $t$ is even, let

$$b_{t+1}^{(i)} = \begin{cases} \mathcal{M} - i, & \text{if } \mathcal{M} - i \geq b_{t-1} + 1; \\ \mathcal{M} - i - 1, & \text{otherwise}. \end{cases}$$

So $b_{t+1}^{(i)} \neq b_{t-1}$ and $b_{t+1}^{(i)} \leq \mathcal{M}$, if $t$ is even.

If $t$ is odd, let

$$b_{t+1}^{(i)} = \begin{cases} i, & \text{if } i \leq b_{t-1} - 1; \\ i + 1, & \text{otherwise}. \end{cases}$$

So $b_{t+1}^{(i)} \neq b_{t-1}$ and $b_{t+1}^{(i)} \leq i + 1 \leq d(u) - 1 \leq \Delta(T) - 1 \leq \mathcal{M} - 1$, if $t$ is odd.

Clearly, all the labels satisfy conditions (S1), (S2) and (S4) for an $[\mathcal{M} + 1]$-sequence. We check (S3). As $d_i \leq h_i(u)$, we have

$$\max\{d_i + i\} \leq \max\{h_i(u) + i\} \leq m(u) \leq \mathcal{M}.$$
Hence, \( d_i \leq \mathcal{M} - i \) for every \( i \). Let \( w \) be a vertex adjacent to \( u_i \) and labelled by \((b_0, b_1, \cdots, b_t, b_{t+1}, b_{t+2})\). If \( t \) is even, then \( b_{t+2} \leq d_i - 1 \leq \mathcal{M} - i - 1 \leq b_{t+1}^{(i)} \). If \( t \) is odd, then \( b_{t+2} \geq \mathcal{M} - 1 - (d_i - 2) \geq i + 1 \geq b_{t+1}^{(i)} \). This verifies (S3). Hence, the mapping by the labelling is well-defined. Moreover, by definition it is clear that the mapping is injective and edge-preserving. This completes the proof. \( \square \)

## 3 Key Lemma

The aim of this section is to establish Lemma 10, which will be used in the next section. Let \( T \) be a tree with maximum degree \( \Delta \) and maximum ordering-degree \( \mathcal{M}(T) > \Delta \). Let \( v \) be a vertex in \( T \) with \( m(v) > \Delta \) and let the ND-sequence of \( v \) be \((h_0, h_1, \cdots, h_{n-1})\), where \( n = \text{deg}(v) \). In the following we define a recursive operation, by first lining up the neighbors of \( v \) by \( N(v) = (v_0, v_1, \cdots, v_{n-1}) \) where \( d(v_i) = h_i \).

The recursive process begins with two lists, \((v_0, v_1, \cdots, v_{n-1})\) and \((h_0, h_1, \cdots, h_{n-1})\). In each step we delete one vertex \( v_t \) and its corresponding \( h_t \) from the lists, where the value of \( t \) is determined by the function \( \sigma \) introduced below. The new list of \( h_i \)'s remains in non-increasing order.

Here is the precise definition of the process. Initially, let \( v^{(0)}_i = v_i \) and \( h^{(0)}_i = h_i \) for \( 0 \leq i \leq n - 1 \), and let \( m^{(0)}(v) = m(v) \).

For \( q \geq 1 \), if \( m^{(q-1)}(v) \leq \Delta \) then stop; else \( m^{(q-1)}(v) > \Delta \) then define

\[
\sigma(q) = \min\{i \mid h^{(q-1)}_i + i = m^{(q-1)}(v), 0 \leq i \leq n - q\},
\]

and

\[
\begin{align*}
&v^{(q)}_i = v^{(q-1)}_{\sigma(q)} \
&h^{(q)}_i = h^{(q-1)}_{\sigma(q)} \\
&v^{(q)}_i = v^{(q-1)}_i \
&h^{(q)}_i = h^{(q-1)}_i, \quad \text{if } 0 \leq i \leq \sigma(q) - 1, \\
&v^{(q)}_i = v^{(q-1)}_{i+1} \
&h^{(q)}_i = h^{(q-1)}_{i+1}, \quad \text{if } \sigma(q) \leq i \leq n - q - 1, \\
&m^{(q)}(v) = \max\{h^{(q)}_i + i \mid 0 \leq i \leq n - q - 1\}.
\end{align*}
\]

In the following, we remark some properties for the above recursive process. In particular, we show that the process stops at some point.

**Proposition 8** Let \( T \) be a tree with maximum degree \( \Delta < \mathcal{M}(T) \). Let \( v \) be a vertex of degree \( n \) and \( m(v) > \Delta \). Let \( p = m(v) - \Delta \). For the process defined above, the following hold for any \( 1 \leq q \leq p \).

1. \( \sigma(q) \geq 1 \).
2. \( h^{(q)}_0, h^{(q)}_2, \ldots, h^{(q)}_{n-q-1} \) is a non-increasing sequence.
(3) \( h^{(q-1)}_{\sigma(q)-1} = h^{(q-1)}_{\sigma(q)} \).

(4) \( m^{(q)}(v) = m^{(q-1)}(v) - 1 = m(v) - q \). Consequently, the process stops at \( q = m(v) - \Delta = p \).

(5) \( \sigma(q) < \sigma(q-1) \). Consequently, \( h^{(q)} \geq h^{(q-1)} \) for \( 2 \leq q \leq p \).

(6) \( v_i^{(q)} = v_i \) for all \( 0 \leq i \leq \sigma(q) - 1 \).

**Proof.** To show (1), suppose \( \sigma(q) = 0 \) for some \( 1 \leq q \leq p - 1 \). Then by definition, \( m^{(q-1)}(v) = h_0^{(q-1)} + 0 = h_0^{(q-1)} \leq \Delta \), a contradiction.

(2) follows since initially \( (h_0, h_1, \ldots, h_{n-1}) \) is in a non-increasing order, and the ordering is kept in each step.

To show (3), by (1) and (2) we get \( h_{\sigma(q)-1}^{(q-1)} \geq h_{\sigma(q)}^{(q-1)} \). Assume to the contrary, \( h_{\sigma(q)-1}^{(q-1)} > h_{\sigma(q)}^{(q-1)} \). Then \( h_{\sigma(q)-1}^{(q-1)} + \sigma(q) - 1 > h_{\sigma(q)}^{(q-1)} + \sigma(q) \), contradicting the choice of \( \sigma(q) \).

To show (4), let \( m^{(q-1)}(v) = x \) and \( m^{(q)}(v) = x' \). By definition, \( x' \leq x - 1 \). By (3), \( x' \geq x - 1 \). Finally, (5) follows by (3) and (4); (6) follows by (5) and the definition of \( v^{(q)} \).

The next result follows directly from the definition of an \( L(j,k) \)-labelling.

**Lemma 9** Let \( v \) be a degree \( n \) vertex in \( G \). If \( f \) is an \( L(j,k) \)-labelling for \( G \) such that \( f(w) < f(v) < f(u) \) for some \( w, u \in N(v) \), then \( sp(f) \geq 2j + (n - 2)k \).

**Lemma 10** Let \( T \) be a tree with maximum degree \( \Delta \). Suppose \( v \) is a vertex of \( T \) with degree \( n \), \( m(v) = M(T) \), and \( d(v,u) \leq 2 \) for all vertex \( u \neq v \) in \( T \). Let the ND-sequence of \( v \) be \( (h_0, h_1, h_2, \ldots, h_{n-1}) \), \( p = m(v) - \Delta \), and let \( h^{(q)} \), \( 1 \leq q \leq p \), be defined as in the above process. Then all the following hold.

(a) \( \lambda_{j,k}(T) \leq j + (m(v) - 1)k \) for all \( j \geq k \).

(b) If \( m(v) = \Delta \), then \( \lambda_{j,k}(T) = j + (\Delta - 1)k \).

(c) If \( m(v) > \Delta \), then \( \lambda_{j,k}(T) = j + (m(v) - 1)k \) if \( j \geq \Delta k \); otherwise,

\[
\lambda_{j,k}(T) \geq \begin{cases} 
\min\{j + (m(v) - 1)k, 2j + (\min\{n, h^{(1)}\} - 2k), \\
\quad \quad \text{if } (m(v) - h^{(1)})k \leq j < \Delta k; \\
\min\{j + (m(v) - q - 1)k, 2j + (\min\{n, h^{(q+1)}\} - 2k), \\
\quad \quad \text{if } (m(v) - h^{(q+1)} - q)k \leq j < (m(v) - h^{(q)} - q + 1)k; \\
\quad \quad \text{for some } 1 \leq q \leq p - 1; \\
\min\{j + (\Delta - 1)k, 2j + (n - 2)k\}, \quad \text{if } j \leq (\Delta - h^{(p)} + 1)k; \\
\end{cases}
\]
(d) If \( n \geq h^{(i)} \) for some \( i \geq 1 \), then the conclusion (in-equality) in (c) becomes an equality for \( j \geq (m(v) - h^{(i)} - i + 1)k \). In particular, if \( n \geq h^{(p)} \) then the above determines the value of \( \lambda_{j,k}(T) \) for \( j \geq (\Delta - h^{(p)} + 1)k \).

Proof. Let \( N(v) = \{v_0, v_1, \ldots, v_{n-1}\} \) with \( d(v_i) = h_i \) and \( N(v_i) = \{v, u_{(i,1)}, u_{(i,2)}, \ldots, u_{(i,h_i-1)}\} \) for \( 0 \leq i \leq n - 1 \). To show (a), we give an \( L(j,k) \)-labelling \( g \) for \( T \) by:

\[
g(v) = j + (m(v) - 1)k, \quad g(v_i) = ik \quad \text{for} \quad 0 \leq i \leq n - 1, \quad \text{and} \quad g(u_{(i,d)}) = j + (m(v) - 1 - d)k.
\]

It is easy to see that \( g \) is an \( L(j,k) \)-labelling for \( T \).

If \( m(v) = \Delta \), then (b) follows from Proposition 2 and (a).

Suppose \( m(v) > \Delta \). Let \( f \) be an optimal \( L(j,k) \)-labelling of \( T \). Since all the neighbors of \( v \) must receive different labels, we can line up the neighbors of \( v \) by \( u_0, u_1, \ldots, u_{n-1} \) so that their labels are in an increasing order. That is,

\[
f(u_i) \geq f(u_{i-1}) + k \quad \text{for} \quad 1 \leq i \leq n - 1.
\]

Let \( d_i \) denote the degree of \( u_i, \) \( 0 \leq i \leq n - 1 \). (Note, \( (d_0, d_1, \ldots, d_{n-1}) \) is not necessarily an ND-sequence.) Hence, \( f(u_i) \geq f(u_0) + ik \geq ik \) for all \( i = 0, 1, \ldots, n - 1 \).

A vertex \( w \) is called straight if \( f(w) > f(w') \) for all \( w' \in N(w) \), or \( f(w) < f(w') \) for all \( w' \in N(v) \). There are three cases to consider.

Case 1. \( v \) is not straight. By Lemma 9,

\[
sp(f) \geq 2j + (n - 2)k. \quad (3.1)
\]

Case 2. Every \( x \in N[v] \) is straight. We claim:

\[
sp(f) \geq j + (m(v) - 1)k. \quad (3.2)
\]

By symmetry, we assume without loss of generality \( f(v) > f(u_i) \) for all \( 0 \leq i \leq n - 1 \). For any \( 0 \leq i \leq n - 1 \), since \( u_i \) is straight and \( v \in N(u_i) \), we get \( f(u_i) < f(z) \) for all \( z \in N(u_i) \). For \( 0 \leq i \leq n - 1 \), let \( f_i = \max\{f(z) \mid z \in N(u_i)\} \). Then,

\[
f_i \geq j + (d_i - 1)k + f(u_i) \geq j + (d_i - 1 + i)k.
\]

So,

\[
sp(f) \geq \max\{f_i \mid 0 \leq i \leq n - 1\}
\geq j + (\max\{d_i + i \mid 0 \leq i \leq n - 1\} - 1)k
\geq j + (m(v) - 1)k.
\]

The last inequality follows by Lemma 5. This proves (3.2).
Case 3. $v$ is straight, but $u_i$ is not for some $i$. We claim (3.3) or (3.4) holds:

$$sp(f) \geq \max\{2j, j + (m(v) - 1)k\},$$

(3.3)

$$sp(f) \geq \min_{1 \leq i \leq p}\{\max\{2j + (h^{(i)} - 2)k, j + (m^{(i)}(v) - 1)k\}\}.$$  

(3.4)

Partition $N(v)$ into two parts, $A$ and $S$, where $S$ consists of all the straight vertices in $N(v)$, and $A = N(v) - S$. Then $A \neq \emptyset$. Let $A = \{u_1, u_2, \ldots, u_x\}$ and $d(u_i) = t_i$ for $1 \leq i \leq x$. Let $T_A$ and $T_S$ be two induced subtrees of $T$ where $V(T_A) = \cup_{w \in S} N[w]$ and $V(T_S) = \cup_{w \in S} N[w]$. Note, $v \in V(T_A) \cap V(T_S)$. Let $g_1$ and $g_2$ be the restrictions of $f$ to $V(T_A)$ and $V(T_S)$, respectively. Then, $sp(f) \geq \max\{sp(g_1), sp(g_2)\}$.

Let $h^* = \max\{t_1, t_2, \ldots, t_x\}$. By Lemma 9, $sp(g_1) \geq 2j + (h^* - 2)k$. For $T_S$, by the same discussion in Case 2, $sp(g_2) \geq j + (m'(v) - 1)k$, where $m'(v)$ is the maximum ordering-degree of $v$ restricted to the subtree $T_S$. Hence,

$$sp(f) \geq \max\{2j + (h^* - 2)k, j + (m'(v) - 1)k\}.$$  

(3.5)

First, assume $m'(v) \geq \Delta$. Let $q = m(v) - m'(v)$. So, $0 \leq q \leq m(v) - \Delta$. If $q = 0$, as $A \neq \emptyset$, we have $h^* - 2 \geq 0$, hence (3.3) holds. Suppose $q \geq 1$. By Proposition 8 (4), $m^{(q)}(v) = m(v) - q = m'(v)$. Therefore, to prove (3.4), by (3.5) it suffices to verify the following:

$$\max\{2j + (h^* - 2)k, j + (m'(v) - 1)k\} \geq \max\{2j + (h^{(q)} - 2)k, j + (m^{(q)}(v) - 1)k\},$$

which is equivalent to $h^* \geq h^{(q)}$. Assume to the contrary, $h^* < h^{(q)}$. By Proposition 8 (5)(6), $v^{(q)} = v^{(q-1)} = v^{(q)}$. Hence, $h^{(q)} = h^{(q-1)} = h^{(q)}$. Since $h^*$ is the largest degree in $V(T_A) - \{v\}$ and as $h^* < h^{(q)}$, we conclude that the vertices $v_i$ with $0 \leq i \leq \sigma(q)$ are all in $T_S$. This implies that $m'(v) \geq h^{(q)} + \sigma(q) = h^{(q-1)} + \sigma(q) = m^{(q-1)}(v) = m^{(q)}(v) + 1 > m^{(q)}(v)$, a contradiction. Hence, (3.4) holds.

Next, assume $m'(v) < \Delta$. Since $f$ is optimal, $sp(f) \geq j + (\Delta - 1)k$. We can replace $m'(v)$ in (3.5) by $\Delta$. Let $q = m(v) - \Delta$. A similar proof as the above will lead to the same conclusion (where $m^{(q)}(v) = m(v) - q = \Delta > m'(v)$).

Now, to get a lower bound for $\lambda_{j,k}(T)$, it suffices to get the least bound among the ones in (3.1), (3.2), (3.3), and (3.4). Notice that (3.2) is weaker than (3.3), so we shall only consider (3.1), (3.2) and (3.4).

Assume $j \geq \Delta k$. Then $j + (m(v) - 1)k \leq 2j + (h^{(1)} - 2)k$, since by definition $h^{(1)} + n - 1 \geq h^{(1)} + \sigma(1) = m(v)$ and because $n \leq \Delta$. By the fact that $h^{(1)} \leq$
\[ h^{(2)} \leq \cdots \leq h^{(p)}, \] we conclude that the minimum among (3.1), (3.2), and (3.4) gives \( \lambda_{j,k}(T) \geq j + (m(v) - 1)k \), for \( j \geq \Delta k \). Hence, by (a), \( \lambda_{j,k}(T) = j + (m(v) - 1)k \), for \( j \geq \Delta k \).

Assume \( j < \Delta k \). By Proposition 8, we have
\[
j + (m(v) - 1)k > j + (m^{(1)}(v) - 1)k > \cdots > j + (m^{(p)}(v) - 1)k = j + (\Delta - 1)k, \quad \text{and}
2j + (h^{(1)} - 2)k \leq 2j + (h^{(2)} - 2)k \leq \cdots \leq 2j + (h^{(p)} - 2)k.
\] (3.6)

To find the minimum among (3.1), (3.2) and (3.4), we consider different values of \( j \).
Notice that
\[
2j + (h^{(x)}(v) - 2)k \geq j + (m^{(x)}(v) - 1)k \iff j \geq (m^{(x-1)}(v) - h^{(x)}(v))k.
\] (3.7)

Assume \( (m(v) - h^{(1)})k \leq j < \Delta k \). Then
\[
\max\{2j + (h^{(1)} - 2)k, j + (m(v) - 2)k\} = 2j + (h^{(1)} - 2)k.
\]

Therefore the least bound among (3.1), (3.2) and (3.4) gives
\[
\lambda_{j,k}(T) \geq \min\{j + (m(v) - 1)k, 2j + (h^{(1)} - 2)k, 2j + (n - 2)k\}.
\]

Hence, the result for the case \( (m(v) - h^{(1)})k \leq j < \Delta k \) follows. Similarly, by (3.6) and (3.7) the remaining cases in (c) can be obtained; we should leave the details to the reader.

Now we prove (d). Assume \( j \geq (m(v) - h^{(q)} - q + 1)k \), and \( n \geq h^{(q)} \) for some \( 1 \leq q \leq p \). Then \( 2j + (h^{(q)} - 2)k \geq j + (m^{(q)} - 1)k \), and \( n \geq h^{(x)} \) for \( 1 \leq x \leq q \). For any \( 1 \leq x \leq q \), let
\[
Q_x = \max\{2j + (h^{(x)} - 2)k, j + (m^{(x)}(v) - 1)k\}.
\]

To prove (d), it suffices to find \( L(j,k) \)-labellings \( g \) and \( f \) for \( T \) with spans \( Q_x \) and \( j + (m(v) - 1)k \), respectively.

Let \( g(v) = Q_x \). For \( 0 \leq i \leq n - x - 1 \), let \( g(v_i) = ik \) and \( g(u_{(i,d)}) = j + (m^{(x)}(v) - 1 - d)k \), where \( u_{(i,d)} \in N(v_i) - \{v\} \) and \( 1 \leq d \leq h^{(x)}_i - 1 \). For \( 1 \leq i \leq x \), let \( g(v^{(i)}) = Q_x - j - (i - 1)k \) and \( g(w^{(i)}_d) = Q_x - dk \) for \( d \leq i - 1 \) or \( g(w^{(i)}_d) = Q_x - 2j - (d - 1)k \) for \( d \geq i \), where \( w^{(i)}_d \in N(v^{(i)}) - \{v\} \) and \( 1 \leq d \leq h^{(i)} - 1 \).

Let \( f(v) = j + (m(v) - 1)k \). For \( 0 \leq i \leq n - 1 \), let \( f(v_i) = ik \) and label the unlabelled neighbors of \( v_i \) by \( j + (m(v) - 1 - l)k, l = 1, \ldots, h_i - 1 \).

It is easy to see that both \( g \) and \( f \) are \( L(j,k) \)-labellings for \( T \), with the desired spans. This completes the proof.
4 Lower Bound and Large Values of $j$

We use the maximum ordering-degree of a tree $T$ to develop a lower bound of $\lambda_{j,k}(T)$ for $j \geq \Delta(T)k$. Moreover, we completely determine the value of $\lambda_{j,k}(T)$ for trees $T$ with $j \geq M(T)k$.

**Theorem 11** Let $T$ be a tree with maximum degree $\Delta$ and maximum ordering-degree $M$. If $j \geq \Delta k$, then $\lambda_{j,k}(T) \geq j + (M - 1)k$.

**Proof.** If $M = \Delta$, the result holds by Proposition 2. Assume $M > \Delta$. Let $v$ be a vertex with degree $n$ and $m(v) = M$. Let $H$ be the subtree of $T$ induced by $v$ and all the vertices within distance 2 from $v$. Since $j \geq \Delta k$, by Lemma 10, we have $\lambda_{j,k}(H) = j + (M - 1)k$. Hence $\lambda_{j,k}(T) \geq j + (M - 1)k$. □

By Theorems 4, 7 and 11, we obtain

**Corollary 12** Let $T$ be a tree with maximum degree $\Delta$ and maximum ordering-degree $M$. Assume $j$ and $k$ are integers with $j \geq \Delta k$. Then $j + (M - 1)k \leq \lambda_{j,k}(T) \leq j + Mk$. Moreover, if $T$ is a subtree of $T_M$, then $\lambda_{j,k}(T) = j + (M - 1)k$.

It was proved by Georges and Mauro [8] that for any graph $G$, $\lambda_{j,k}(G) = \alpha j + \beta k$ for some non-negative integers $\alpha$ and $\beta$. By Corollary 12, if $T$ is a tree of maximum degree $\Delta$ and maximum ordering-degree $M$, and if $j \geq \Delta k$, then there only three possible values for $\lambda_{j,k}(T)$. Precisely, if $j \geq \Delta k$, then

$$\lambda_{j,k}(T) \in \{j + (M - 1)k, \beta k, j + Mk\},$$

where $M$ is the maximum ordering-degree of $T$, and $\beta$ is a non-negative integer with $j + (M - 1)k \leq \beta k \leq j + Mk$.

**Theorem 13** Let $j, k, M$ be positive integers with $j \geq Mk$. For any graph $G$, the following are equivalent:

(1) $\lambda_{j,k}(G) < j + Mk$.

(2) There exists a $\lambda_{j,k}$-labelling $g$ for $G$ such that for any vertex $v$ in $G$, $g(v)$ is of the form $a_v j + b_v k$ with $a_v \in \{0, 1\}$ and $b_v \in \{0, 1, \ldots, M - 1\}$. Moreover, the following hold:
(D1) If \( d_G(u, v) = 1 \), then \( a_u \neq a_v \). If \( a_u = 0 \) and \( a_v = 1 \), then \( b_u \leq b_v \).

(D2) If \( d_G(u, v) = 2 \), then \( a_u = a_v \) and \( b_u \neq b_v \).

(3) \( \lambda_{j,k}(G) \leq j + (M - 1)k \).

**Proof.** It is enough to show (1) \( \Rightarrow \) (2). Assume \( \lambda_{j,k}(G) < j + Mk \). Let \( f \) be a \( \lambda_{j,k} \)-labelling for \( G \). For every vertex \( v \), write \( f(v) = a_v j + b_v k + r_v \), where \( a_v, b_v, r_v \) are non-negative integers satisfying \( 0 \leq b_v k + r_v < j \) and \( 0 \leq r_v < k \).

Since \( j \geq Mk \) and \( \lambda_{j,k}(G) < j + Mk \), we have \( a_v = 0 \) or \( 1 \). We claim that \( 0 \leq b_v \leq M - 1 \). If \( a_v = 1 \), this is obviously true. Assume \( a_v = 0 \). Choose a vertex \( u \) adjacent to \( v \). (If \( v \) is isolated, let \( f(v) = 0 \).) By the assumption that \( f(v) = b_v k < j \), it must be \( f(v) < f(u) \), and so \( f(v) \leq f(u) - j < Mk \); implying \( b_v \leq M - 1 \).

Define a function \( g \) on \( V(G) \) by \( g(v) = a_v j + b_v k \). It suffices to show that \( g \) is an \( L(j,k) \)-labelling for \( G \), and \( g \) satisfies (D1) and (D2). Suppose \( u \) and \( v \) are adjacent vertices in \( G \). As \( |f(u) - f(v)| \geq j \geq Mk \), it must be the case that \( a_u \neq a_v \).

Without loss of generality, assume \( a_u = 0 \) and \( a_v = 1 \). Then \( f(u) < f(v) \). So we have \( f(u) \leq f(v) - j \), implying \( b_u k + r_u \leq b_v k + r_v \). As \( b_u k \leq b_u k + r_u \leq b_v k + r_v < (b_v + 1)k \), we get \( b_u \leq b_v \). Hence, \( g(u) \leq g(v) - j \), and (D1) holds.

Next, suppose \( d_G(u, v) = 2 \). Choose a vertex \( w \) adjacent to both \( u \) and \( v \). Then \( a_u \neq a_w \) and \( a_w \neq a_v \), implying \( a_u = a_v \). As \( |f(u) - f(v)| \geq k \), we have \( b_u \neq b_v \) (so (D2) holds) and \( |g(u) - g(v)| \geq k \). This verifies that \( g \) is a \( \lambda_{j,k} \)-labelling for \( G \), which satisfies (D1) and (D2).

Let \( T \) be a tree with maximum ordering-degree \( M \). Let \( j \geq Mk \). Because \( M(T) \geq \Delta(T) \), by Theorem 13 and Corollary 12, we have \( \lambda_{j,k}(T) \in \{j + (M - 1)k, j + Mk\} \). Indeed, the value of \( \lambda_{j,k}(T) \) can be completely settled as follows.

**Theorem 14** Let \( T \) be a tree with maximum ordering-degree \( M \). If \( j \geq Mk \), then

\[
\lambda_{j,k}(T) = \begin{cases} 
    j + (M - 1)k, & \text{if } T \text{ is a subtree of } T_M; \\
    j + Mk, & \text{otherwise.}
\end{cases}
\]

**Proof.** Assume \( j \geq Mk \). By Theorem 13 and Corollary 12, it is enough to show that if \( \lambda_{j,k}(T) = j + (M - 1)k \), then \( T \) is a subtree of \( T_M \).

Suppose \( \lambda_{j,k}(T) = j + (M - 1)k \). By Theorem 13, \( T \) has a \( \lambda_{j,k} \)-labelling \( g \) such that for any vertex \( v \), \( g(v) = a_v j + b_v k \) where \( a_v \in \{0, 1\} \) and \( b_v \in \{0, 1, \ldots, M - 1\} \) satisfying conditions (D1) and (D2).
Let $v$ be a vertex with $m(v) = M$. Since $g$ is a $\lambda_{j,k}$-labelling, there exists a vertex $v_0$ such that $g(v_0) = 0$. To prove that $T$ is a subtree of $T_M$, it suffices to find an injective homomorphism from $T$ to $T_M$.

For any vertex $u$ in $T$, there is a unique path from $v_0$ to $u$. Denote this path by $P : v_0, v_1, \ldots, v_m = u$. According to (D1) and (D2), $(b_{v_0}, b_{v_1}, \ldots, b_{v_m})$ is an $[M]$-sequence. Define a function $\phi : V(T) \to V(T_M)$ by $\phi(u) = (b_{v_0}, b_{v_1}, \ldots, b_{v_m})$. Clearly, $\phi$ is an injective homomorphism. Hence $T$ is a subtree of $T_M$. □

References


