On \((d, 1)\)-Total Numbers of Graphs

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Abstract

A \((d, 1)\)-total labelling of a graph \(G\) assigns integers to the vertices and edges of \(G\) such that adjacent vertices receive distinct labels, adjacent edges receive distinct labels, and a vertex and its incident edges receive labels that differ in absolute value by at least \(d\). The span of a \((d, 1)\)-total labelling is the maximum difference between two labels. The \((d, 1)\)-total number, denoted \(\lambda_T^{(d)}(G)\), is defined to be the least span among all \((d, 1)\)-total labellings of \(G\). We prove new upper bounds for \(\lambda_T^{(d)}(G)\), compute some \(\lambda_T^{(d)}(K_{m,n})\) for complete bipartite graphs \(K_{m,n}\), and completely determine all \(\lambda_T^{(d)}(K_{m,n})\) for \(d = 1, 2, 3\). We also propose a conjecture on an upper bound for \(\lambda_T^{(d)}(G)\) in terms of the chromatic number and the chromatic index of \(G\).

Key words: channel assignment, \(L(2, 1)\)-labelling, \((d, 1)\)-total labelling, chromatic number, chromatic index

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1 Introduction

Let $d$ be a positive integer and $G(V, E)$ be a finite graph without loops or multiple edges. We always assume that $G$ has at least one edge without explicitly saying so. A $(d, 1)$-total labelling of $G$ is an integer-valued function $f$ defined on the set $V(G) \cup E(G)$ such that

$$|f(x) - f(y)| \begin{cases} 1 & \text{if vertices } x \text{ and } y \text{ are adjacent;} \\ 1 & \text{if edges } x \text{ and } y \text{ are adjacent;} \\ d & \text{if vertex } x \text{ and edge } y \text{ are incident.} \end{cases}$$

We may require $|f(x) - f(y)|$, for adjacent elements $x$ and $y$, be greater than or equal to $s$, instead of 1, in the above defining inequality for some given positive integer $s$ to get a more general notion of a $(d, s)$-total labelling; nevertheless we concentrate our attention only to the special case $s = 1$ in this paper. A $(d, 1)$-total labelling taking values in the set $\{0, 1, \ldots, k\}$ is called a $[k]$-$(d, 1)$-total labelling. The span of a $(d, 1)$-total labelling is the maximum difference between two labels. The minimum span, i.e. the minimum $k$, among all $[k]$-$(d, 1)$-total labellings of $G$, denoted $\lambda^T_d(G)$, is called the $(d, 1)$-total number of $G$.

A $(d, 1)$-total labelling of $G$ is a generalization of an $L(2, 1)$-labelling of the subdivision of $G$ studied in Whittlesey, Georges, and Mauro [15]. The notion of an $L(2, 1)$-labelling was motivated by an interference avoidance problem, introduced in Hale [7], in the assignment of radio frequency bands to transmitters. An $L(2, 1)$-labelling of $G$ assigns nonnegative integer labels to the vertices of $G$ so that vertices at distance two receive distinct labels and adjacent vertices receive labels that differ in absolute value by at least 2. Griggs and Yeh [6] initiated a systematic study into $L(2, 1)$-labellings of graphs that has been intensively developed ever since. The reader is referred to Yeh [17] for a recent survey of results and generalizations of $L(2, 1)$-labellings. The subdivision $G^S$ of a graph $G$ is the graph obtained by inserting one new vertex to each of the edges of $G$. If we define the span of an $L(2, 1)$-labelling to be the maximum difference between two labels, then the minimum span among all $L(2, 1)$-labellings of $G^S$ is precisely $\lambda^T_2(G)$.

Havet and Yu [8] first introduced the notion of a $(d, 1)$-total labelling and their results have published only recently in [9]. Let $\Delta(G)$ denote the maximum degree of $G$. Havet and Yu proposed the following conjecture.

$(d, 1)$-Total Labelling Conjecture. $\lambda^T_d(G) \leq \min\{\Delta(G) + 2d - 1, 2\Delta(G) + d - 1\}$.
In addition to [9], positive evidence to this conjecture has also been given in [1], [4], and [13]. Note that $\lambda_d^T(G) + 1$ is equal to the total chromatic number $\chi''(G)$ of the graph $G$ and the $(d,1)$-total labelling conjecture for the case $d = 1$ is equivalent to the well-known Total Coloring Conjecture proposed by Behzad [2] and independently by Vizing [14].

It should be pointed out that a $(d,1)$-total labelling is a special case ($r = s = 1$) of an $[r,s,d]$-coloring introduced and studied in [10], [11], and [12]. The $\lambda_{d,1,1}(G)$ of a graph $G$ defined there is exactly $\lambda_d^T(G) + 1$.

In Section 2, we will derive upper bounds for $\lambda_d^T(G)$. Based on these values, we propose an upper bound conjecture in terms of the chromatic number and the chromatic index of $G$.

In Section 3, we compute some values of $\lambda_d^T(K_{m,n})$ for complete bipartite graphs $K_{m,n}$ and completely determine all $\lambda_d^T(K_{m,n})$ for $d = 1, 2, 3$. These values give further support to the $(d,1)$-total labelling conjecture.

2 Upper Bounds

We are going to present upper bounds for $\lambda_d^T(G)$ in terms of its maximum degree $\Delta(G)$, chromatic number $\chi(G)$, chromatic index $\chi'(G)$, and list chromatic index $\chi'_l(G)$. We will propose a conjecture on an upper bound of $\lambda_d^T(G)$ at the end of this section.

Let $\chi(G)$, or $\chi'(G)$, denote the smallest number of colors needed to color the vertices, respectively the edges, of $G$ so that adjacent elements receive distinct colors. A vertex-coloring or an edge-coloring satisfying the above condition is said to be a proper vertex-coloring or edge-coloring. If each edge $e$ of $G$ is assigned a list $L(e)$ of possible colors and $G$ has a proper edge-coloring $\phi$ such that $\phi(e) \in L(e)$ for all $e \in E(G)$, then we say that $G$ is $L$-edge-colorable. The graph $G$ is said to be $k$-edge-choosable if it is $L$-edge-colorable for every assignment $L$ satisfying $|L(e)| = k$ for all $e \in E(G)$. Let $\chi'_l(G)$ denote the smallest $k$ such that $G$ is $k$-edge-choosable.

The following two lemmas were proved in Havet and Yu [9] and the case for $d = 2$ first appeared in Whittlesey, Georges, and Mauro [15].

**Lemma 1** For any graph $G$, $\lambda_d^T(G) \leq \chi(G) + \chi'(G) + d - 2$.

**Lemma 2** For any graph $G$, $\lambda_d^T(G) \leq 2\Delta(G) + d - 1$.

Throughout this paper, a proper vertex-coloring, or edge-coloring, using colors from the set $\{0,1,\ldots,k-1\}$ is said to be a $k$-vertex-coloring, or $k$-edge-
For integers $a \leq b$, we use $[a, b]$ to denote the set \{a, a + 1, \ldots, b\}. For integers $a$ and $d$, the set $[a - d + 1, a + d - 1]$ is denote by $[a]_d$.

**Theorem 3** For any graph $G$, $\lambda^T_d(G) \leq \chi'_d(G) + 4d - 3$.

**Proof.** Since $\chi(G) \leq \Delta(G) + 1$, we can give a proper vertex-coloring $f_1$ for $G$ using colors $0, 1, \ldots, \Delta(G)$. For each edge $e = xy$, we define the list

$$L(e) = [0, \chi'_d(G) + 4d - 3] \setminus ([f_1(x)]_d \cup [f_1(y)]_d).$$

As $|L(e)| \geq \chi'_d(G)$, there exists an $L$-coloring $f_2$ for the edges of $G$. Since $\chi'_d(G) \geq \chi'_d(G) \geq \Delta(G)$, we have $\chi'_d(G) + 4d - 3 \geq \Delta(G)$. Consequently, $f_1 \cup f_2$ forms a $[\chi'_d(G) + 4d - 3]-(d, 1)$-total labelling of $G$.

Borodin, Kostochka, and Woodall [3] proved that $\chi'_d(G) \leq [\frac{3}{2}\Delta(G)]$ for a multigraph graph $G$. Hence, by Theorem 3, the following upper bound for $\lambda^T_d(G)$ emerges.

**Theorem 4** For any graph $G$, $\lambda^T_d(G) \leq [\frac{3}{2}\Delta(G)] + 4d - 3$.

Note that, for fixed $d$ and sufficient large $\Delta(G)$, the upper bound for $\lambda^T_d(G)$ in Theorem 4 is better than the one in Lemma 2. In the rest of this section, we shall improve the bounds of Lemmas 1 and 2.

**Theorem 5** Let $G$ be a graph with $\chi(G) = k$ and $\chi'(G) = k'$. If $k \geq 3d$, then $\lambda^T_d(G) \leq s + k' - 1$, where $s$ is equal to $4d - 2$ when $k = 3d$ or $3d + 1$, and equal to $\lceil (k + 9d - 5)/3 \rceil$ when $k \geq 3d + 2$.

**Proof.** We choose a mapping $f : V(G) \cup E(G) \rightarrow [0, s + k' - 1]$ such that the restriction of $f$ to $V(G)$ is a $k$-vertex-coloring and the restriction of $f$ to $E(G)$ is a proper edge-coloring using colors in $[s, s + k' - 1]$.

Let $G'$ be the subgraph of $G$ induced by the edges in $E' = \{e \in E(G) \mid f(e) \in [s, k + d - 2]\}$. Then $\Delta(G') \leq k + d - s - 1$. To any $e = xy \in E(G')$, we assign the list $L(e) = [0, k + d - 2] \setminus ([f(x)]_d \cup [f(y)]_d)$. Then $|L(e)| \geq k - 3d + 1$. Since $\Delta(G') \leq k + d - s - 1$, $G'$ is a disjoint union of edges when $k = 3d$, and is a disjoint union of paths and even cycles when $k = 3d + 1$. It is well-known that $\chi'_d(G') \leq |L(e)|$ in these cases. When $k \geq 3d + 2$, it follows from $s \geq (k + 9d - 5)/3$ that $3(k + d - s - 1)/2 \leq k + 3d + 1$. Since $\chi'_d(G') \leq [3\Delta(G)/2] \leq 3(k + d - s - 1)/2$, we have $\chi'_d(G') \leq |L(e)|$ again. Hence, there always exists an $L$-edge-coloring $f'$ for $G'$. Re-labelling edges in $G'$ by $f'$ while keeping the rest of $G$ unchanged, we get an $[s + k' - 1]-(d, 1)$-total labelling for $G$.

By Theorem 5, the following conjecture holds for graphs with $\chi(G) \geq 3d$. 

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4
Conjecture 1 Let a graph $G$ satisfy $\chi(G) > \max\{2, d\}$. Then

$$\lambda_d^T(G) \leq \chi(G) + \chi'(G) + d - 3.$$ 

The known values of $\lambda_d^T(K_n)$ for complete graphs $K_n$ on $n$ vertices that have been computed in [9] support the above conjecture. The following corollary also appeared in [9].

Corollary 6 Let $G$ be a bipartite graph. Then $\Delta(G) + d - 1 \leq \lambda_d^T(G) \leq \Delta(G) + d$ and $\lambda_d^T(G) = \Delta(G) + d$ when $d \geq \Delta(G)$ or $G$ is regular.

For a bipartite graph $G$, it is well-known that $\chi'(G) = \Delta(G)$. Hence, a consequence of Corollary 6 is $\lambda_d^T(G) = \Delta(G) + d = \chi(G) + \chi'(G) + d - 2$ for a bipartite regular graph $G$. This together with the fact $\lambda_4^T(K_4) = 9$ show that the assumption $\chi(G) > \max\{2, d\}$ in Conjecture 1 cannot be removed.

3 Complete Bipartite Graphs

The following can be easily derived when we examine the label of a vertex of maximum degree and the labels of its incident edges.

Lemma 7 (1) $\lambda_d^T(G) \geq \Delta(G) + d - 1$.

(2) If $\lambda_d^T(G) = \Delta(G) + d - 1$, then each vertex of maximum degree is labelled with 0 or $\Delta(G) + d - 1$ in any $[\Delta(G) + d - 1]-(d,1)$-total labelling.

Throughout this section, let $K_{m,n}$ ($m \geq n$) denote the complete bipartite graph with parts $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. By Corollary 6, $m + d - 1 \leq \lambda_d^T(K_{m,n}) \leq m + d$. When a function $f$ is defined over the edges of $K_{m,n}$, we write $f(i, j)$ for $f(x_iy_j)$. Furthermore, let $X_i = \{f(i, j) \mid 1 \leq j \leq m\}$ and $Y_j = \{f(i, j) \mid 1 \leq i \leq n\}$.

Theorem 8 The following statements are equivalent.

(1) $m \geq \min\{2n, n + 2d - 1\}$ and $m \geq n + d$.

(2) There exists an $[m + d - 1]-(d,1)$-total labelling $f$ for $K_{m,n}$ such that $f(x) = 0$ for all $x \in X$, or $f(x) = m + d - 1$ for all $x \in X$.

Proof. (1) $\Rightarrow$ (2). We are going to construct an $[m + d - 1]-(d,1)$-total labelling $f$ for $K_{m,n}$ such that $f(x) = 0$ for all $x \in X$. 

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First assume that \( m \geq 2n \). Let \( \rho \) be the composition of the two cyclic permutations \((1 2 \cdots n)\) and \((n + 1 \ n + 2 \cdots m)\) on the set \([1, m]\). Let \( f(x_i) = 0 \) for all \( 1 \leq i \leq n \), \( f(y_j) = m + d - 1 \) for \( 1 \leq j \leq n \), \( f(y_j) = 1 \) for \( n + 1 \leq j \leq m \), and \( f(i, j) = (d - 1) + \rho_{d-1}(j) \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Since \( m \geq 2n \), adjacent edges are labelled with distinct labels. We see that \( Y_j = [d, d+n-1] \) when \( 1 \leq j \leq n \) and \( Y_j \subseteq [d+n, d+m-1] \) when \( n < j \leq m \). Since \( 1 \leq d \leq m - n \), the absolute difference between the label of any vertex and the label of any of its incident edge is at least \( d \), hence \( f \) satisfies our requirements.

Next assume that \( m \geq n+2d-1 \). Let \( \sigma \) be the cyclic permutation \((1 2 \cdots m)\) on the set \([1, m]\). For \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), let \( f(x_i) = 0 \), \( f(y_j) = (d-1) + \sigma^{n-1+d}(j) \), and \( f(i, j) = (d-1) + \sigma^{i-1}(j) \). Adjacent edges are obviously labelled with distinct labels. Since \( m \geq n+2d-1 \), we see that \([\sigma^{n-1+d}(j) - \sigma^{i-1}(j)] \geq d\) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), hence \( f \) satisfies our requirements.

\[(2) \Rightarrow (1) \]. Assume there exists an \([m+d-1]-(d,1)\)-total labelling \( f \) for \( K_{m,n} \) such that \( f(x) = 0 \) for all \( x \in X \). (By symmetry, we only need to show this case.)

Since \( f(x_i) = 0 \) for all \( i \), we have \( f(i, j) \geq d \) and \( X_i = [d, d+n-1] \) for all \( i \) and \( j \). Without loss of generality, we may assume that \( d \in Y_j \) for \( 1 \leq j \leq n \), and hence \( f(y_j) \geq 2d \). Let \( t_j \) denote the largest number in \( Y_j \). Then \( t_j \geq n + d - 1 \).

Assume that \( t_p > f(y_p) \) for some \( p \in [1, n] \). Then \([f(y_p)]_d \subseteq [d, t_p]\) and \([f(y_p)]_d \cap Y_p = \emptyset\). Moreover, since \(|[f(y_p)]_d| = 2d-1\), \( Y_p \in [d, t_p]\), and \( |Y_p| = n\), it follows that \( t_p - d - 1 \geq n + 2d - 1 \). As \( t_p \leq m + d - 1 \), we conclude that \( m \geq n + 2d - 1 \geq n + d \).

Assume \( t_j < f(y_j) \) for all \( j \in [1, n] \). Then we have \( t_j \leq f(y_j) - d \leq m - 1 \), implying \( n + d - 1 \leq m - 1 \). Therefore, \( m \geq n + d \). Moreover, it also follows that \( Y_j \subseteq [d, m-1] \) for \( 1 \leq j \leq n \). This implies that the edges that can be assigned labels from the set \([m, m + d - 1]\) must be incident to \( y_j \) for some \( j \in [n+1, m] \). Hence, \( nd = \sum_{j=n+1}^{m} |Y_j \cap [m, m + d - 1]| \leq (m-n)d \), implying \( 2n \leq m \).

By Theorem 8, to further investigate the values of \( m \) and \( n \) such that \( \lambda^*_d(K_{m,n}) = m + d - 1 \), it remains to study the following two possibilities.

**Case 1.** \( m \geq \min\{2n, n + 2d - 1\} \) and \( m < n + d \), or equivalently, \( 2n \leq m < n + d \).

**Case 2.** \( m < \min\{2n, n + 2d - 1\} \).
We shall deal with Cases 1 and 2 in Theorems 9 and 10, respectively. There is one more notation used in the proofs of Theorems 9 and 10. For any \([m+d-1]-(d,1)\)-total labelling \(f\) for \(K_{m,n}\), by Lemma 7, each vertex \(x_i \in X\) is labelled with either 0 or \(m + d - 1\). Denote

\[
I = \{i \mid f(x_i) = 0 \text{ and } 1 \leq i \leq n\}.
\]

Then we have \(X_i = [d, m + d - 1]\) for each \(i \in I\), while \(X_i = [0, m - 1]\) for each \(i \notin I\).

**Theorem 9** If \(2n \leq m < n + d\), then \(\lambda_d^T(K_{m,n}) = m + d\).

**Proof.** The assumption \(2n \leq m < n + d\) implies that \(n < d\) and \(m < 2d\). Suppose to the contrary that \(\lambda_d^T(K_{m,n}) = m + d - 1\). Let \(f\) be an \([m+d-1]-(d,1)\)-total labelling. By Theorem 8, \(1 \leq |I| \leq n - 1\). Since \(d \in X_i\) for any \(i \in I\), we have \(d \in Y_j\) for some \(j\). It implies that \(2d \leq f(y_j) \leq m + d - 2\) because \(f(y_j) \notin \{0, m + d - 1\}\), and hence \(d \leq m - 2\). It follows that \(d \in X_i\) for any \(i \not\in I\). Now \(d\) belongs to all \(X_i\)'s. Without loss of generality, we may assume that \(d \in Y_j\) for \(1 \leq j \leq n\).

Pick \(i_0 \in I\). So \(X_{i_0} = [d, m + d - 1]\). Because all \(y_j\), \(1 \leq j \leq n\), are adjacent to \(x_{i_0}\), there exists \(w \geq n + d - 1\) such that \(w \in Y_{j_0}\) for some \(j_0 \in [1, n]\). We know that \(2d \leq f(y_{j_0}) \leq m + d - 2\). If \(\alpha \in [m - 1, m + d - 1]\), then \(|f(y_{j_0}) - \alpha| < d\) since \(m < 2d\). It follows that \(Y_{j_0} \cap [m - 1, m + d - 1] = \emptyset\) and \(n + d - 1 \leq w \leq m - 2\), contradicting the assumption \(m < n + d\). 

**Theorem 10** Suppose that \(m < \min\{2n, n + 2d - 1\}\) and \(\lambda_d^T(K_{m,n}) = m + d - 1\). Then all the following statements hold.

1. \(m \geq 3d + 1\).
2. \((n - m + 3d - 1)(2n - m) \leq nd\).
3. \(m \geq n + d\).
4. \(\alpha = [(m - d - 2)/(2d - 1)]\).

**Proof.** Assume \(m < \min\{2n, n + 2d - 1\}\) and \(\lambda_d^T(K_{m,n}) = m + d - 1\). Let \(f\) be an \([m+d-1]-(d,1)\)-total labelling. By Theorem 8, \(1 \leq |I| \leq n - 1\). Without loss of generality, we may assume that \(\{d, m - 1\} \subseteq Y_j\) for \(1 \leq j \leq 2n - m\). It follows that \(2d \leq f(y_j) \leq m - d - 1\), and hence \(m \geq 3d + 1\). This completes the proof for (1).

Since \(2d \leq f(y_j) \leq m - d - 1\) for \(1 \leq j \leq 2n - m\), we have \(|[d, m - 1] \cap Y_j| \leq m - 3d + 1\). As \(|Y_j| = n\), it follows that \(|(0, d - 1] \cup [m, m + d - 1]) \cap Y_j| \geq n - m + 3d - 1\). Note that each label in \([0, d - 1]\) is assigned to exactly \(n - |I|\).
edges, while each label in \([m, m + d - 1]\) is assigned to exactly \(|I|\) edges. We conclude that

\[
(n - m + 3d - 1)(2n - m) \leq 2^{n-m} \left| \bigcup_{j=1}^{n-m} \left( [0, d - 1] \cup [m, m + d - 1] \right) \cap Y_j \right| \\
\leq nd.
\]

This completes the proof for (2).

To prove (3), consider \(Y_j\) for \(1 \leq j \leq 2n - m\). Since \(2d \leq f(y_j) \leq m - d - 1\) and \([f(y_j)]_d \cap Y_j = \emptyset\), we obtain that \(n = |Y_j| = |[0, m + d - 1] \cap Y_j| \leq m + d - (2d - 1) = m - d + 1\). Hence \(m \geq n + d - 1\). Suppose \(m = n + d - 1\). Then (2) implies \(n \leq 2d - 2\). This is impossible since \(m = n + d - 1 \geq 3d + 1\) by (1).

It follows from (1) that the number \(\alpha\) in (4) is positive and \(\alpha(2d - 1) + 1 \leq m - d - 1 \leq (\alpha + 1)(2d - 1)\). For each \(j \in [1, 2n - m]\), since \(2d \leq f(y_j) \leq m - d - 1\) and \([f(y_j)]_d \cap Y_j = \emptyset\), the following statement holds: For each \(s \in [1, \alpha]\), if \(f(y_j) \in [s(2d - 1) + 1, (s + 1)(2d - 1)]\), then \(s(2d - 1) + d \notin Y_j\). For each \(i \in [1, \alpha]\), let \(t_i = \{|j| j \in [1, 2n - m] \text{ and } f(y_j) \in [i(2d - 1) + 1, (i + 1)(2d - 1)]\}\). Because \(t_1 + t_2 + \ldots + t_\alpha = 2n - m\), there exists some \(k \in [1, \alpha]\) such that \(t_k \geq (2n - m)/\alpha\). Therefore, \(k(2d - 1) + d\) does not belong to at least \((2n - m)/\alpha\) of the \(Y_j\)'s for \(1 \leq j \leq 2n - m\). Since the label \(k(2d - 1) + d\) belongs to exactly \(n\) of the \(Y_j\)'s for \(1 \leq j \leq m\), we conclude that \((2n - m)/\alpha \leq m - n\), hence (4) follows.

The following is an immediate consequence of Theorem 9 and Theorem 10(3).

**Corollary 11** If \(m < n + d\), then \(\chi^T_\alpha(K_{m,n}) = m + d\).

Now we are ready to give exact values of \(\chi^T_\alpha(K_{m,n})\) for \(d = 1, 2, 3\). The case for \(d = 1\) is completely determined by the total chromatic number of \(K_{m,n}\) and the reader is referred to Theorem 3.2 in Yap [16] for a proof.

**Theorem 12** Let \(1 \leq n \leq m\). Then

\[
\chi^T_1(K_{m,n}) = \chi''(K_{m,n}) - 1 = m + \delta_{m,n},
\]

where \(\delta_{m,n}\) denotes the Kronecker delta, i.e., its value is 1 if \(m = n\) and is 0 otherwise.
**Theorem 13** Let $1 \leq n \leq m$. Then

$$
\lambda^T_2(K_{m,n}) = \begin{cases} 
  m + 2 & \text{if } m \leq n + 1, \text{ or} \\
  m = n + 2 \text{ and } n \geq 3; \\
  m + 1 & \text{otherwise}.
\end{cases}
$$

**Proof.** By Corollary 6, it suffices to consider the case for $m > n$. The results for $m \geq n + 3$ follow from Theorem 8. For $m = n + 1$, the result follows from Corollary 11. Assume $m = n + 2$. The cases for $n = 1, 2$ follow from Theorem 8. The cases for $n = 3, 4$ follow from Theorem 10(1). The cases for $n = 5, 6$ follow from Theorem 10(4). All the remaining cases follow from Theorem 10(2).

**Theorem 14** Let $1 \leq n \leq m$. Then

$$
\lambda^T_3(K_{m,n}) = \begin{cases} 
  m + 3 & \text{if } m \leq n + 2, \text{ or} \\
  m = n + 3 \text{ and } n \geq 4, \text{ or} \\
  m = n + 4 \text{ and } n = 5, 9, 10, 13, 14, 15; \\
  m + 2 & \text{otherwise}.
\end{cases}
$$

**Proof.** By Corollary 6, it suffices to consider the case for $m > n$. The results for $m \leq n + 2$ and $m \geq n + 5$, respectively, follow from Corollary 11 and Theorem 8.

Assume $m = n + 3$. The cases for $n = 1, 2, 3$ follow from Theorem 8. The cases for $n = 4, 5, 6$ follow from Theorem 10(1). The case for $n = 7$ follows from Theorem 10(4). The remaining cases for $n \geq 8$ follow from Theorem 10(2).

Finally assume $m = n + 4$. The cases for $n = 1, 2, 3, 4$ follow from Theorem 8. The case for $n = 5$ follows from Theorem 10(1). The cases for $n \geq 17$ follow from Theorem 10(2). The cases for $n = 9, 10, 13, 14, 15$ follow from Theorem 10(4). In the appendix, we list $[n+6]-(3, 1)$-total labellings obtained by ad hoc methods for each of the $K_{n+4,n}$, $n = 6, 7, 8, 11, 12, 16$.

We conclude this paper with the following problem whose answer is positive for $d = 1, 2, 3$ from our results.

**Problem.** Under the assumption that $m < \min\{2n, n+2d-1\}$, are conditions (1) to (4) in Theorem 10 sufficient for $\lambda^T_d(K_{m,n}) = m + d - 1$?

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A Appendix

When \( n \) is one of the numbers 6, 7, 8, 11, 12, or 16, an \([n + 6]-(3,1)\)-total labelling for \( K_{n+4,n} \) is given below by a table. The notation used is as follows.

The label of the \( i \)-th row is assigned to the vertex \( x_i \in X \).

The label of the \( j \)-th column is assigned to the vertex \( y_j \in Y \).

The label at the \((i, j)\) cell is assigned to the edge \( x_iy_j \).

\[
\begin{array}{cccccccccccc}
\hline
& 6 & 6 & 9 & 9 & 11 & 11 & 1 & 1 & 1 & 1 \\
\hline
12 & 2 & 0 & 1 & 3 & 5 & 4 & 7 & 8 & 9 & 6 \\
12 & 3 & 1 & 0 & 2 & 6 & 5 & 9 & 4 & 8 & 7 \\
12 & 9 & 2 & 3 & 0 & 1 & 6 & 8 & 7 & 4 & 5 \\
0 & 10 & 3 & 4 & 6 & 8 & 7 & 12 & 11 & 5 & 9 \\
0 & 11 & 9 & 5 & 4 & 7 & 3 & 10 & 12 & 6 & 8 \\
0 & 12 & 10 & 6 & 5 & 3 & 8 & 11 & 9 & 7 & 4 \\
\hline
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\hline
& 6 & 6 & 7 & 12 & 12 & 12 & 12 & 1 & 1 & 1 & 1 \\
\hline
13 & 0 & 2 & 1 & 5 & 8 & 3 & 7 & 9 & 10 & 4 & 6 \\
13 & 1 & 0 & 3 & 6 & 2 & 4 & 5 & 8 & 7 & 9 & 10 \\
13 & 3 & 1 & 4 & 2 & 5 & 0 & 6 & 10 & 9 & 8 & 7 \\
0 & 9 & 3 & 11 & 7 & 6 & 5 & 4 & 12 & 8 & 10 & 13 \\
0 & 10 & 9 & 12 & 8 & 7 & 6 & 3 & 11 & 5 & 13 & 4 \\
0 & 12 & 10 & 13 & 3 & 4 & 7 & 8 & 6 & 11 & 5 & 9 \\
0 & 13 & 11 & 10 & 4 & 3 & 8 & 9 & 7 & 12 & 6 & 5 \\
\hline
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\hline
& 6 & 6 & 7 & 7 & 13 & 13 & 13 & 1 & 1 & 1 & 1 \\
\hline
14 & 0 & 1 & 3 & 2 & 5 & 6 & 4 & 8 & 11 & 7 & 9 & 10 \\
14 & 1 & 0 & 2 & 3 & 6 & 7 & 10 & 9 & 5 & 4 & 11 & 8 \\
14 & 3 & 2 & 0 & 1 & 9 & 10 & 6 & 7 & 4 & 8 & 5 & 11 \\
14 & 2 & 3 & 1 & 10 & 0 & 4 & 7 & 6 & 9 & 11 & 8 & 5 \\
0 & 11 & 10 & 4 & 13 & 3 & 8 & 9 & 5 & 12 & 14 & 6 & 7 \\
0 & 12 & 11 & 13 & 14 & 8 & 3 & 5 & 4 & 7 & 9 & 10 & 6 \\
0 & 13 & 12 & 14 & 11 & 7 & 5 & 3 & 10 & 8 & 6 & 4 & 9 \\
0 & 14 & 13 & 11 & 12 & 10 & 9 & 8 & 3 & 6 & 5 & 7 & 4 \\
\hline
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\hline
& 6 & 6 & 11 & 11 & 11 & 11 & 16 & 16 & 16 & 16 & 1 & 1 & 1 & 1 \\
\hline
0 & 11 & 16 & 9 & 5 & 6 & 15 & 17 & 10 & 3 & 8 & 4 & 12 & 13 & 14 & 7 \\
0 & 12 & 11 & 10 & 6 & 14 & 17 & 5 & 13 & 4 & 3 & 9 & 15 & 7 & 16 & 8 \\
0 & 13 & 12 & 17 & 7 & 16 & 3 & 15 & 4 & 5 & 6 & 10 & 8 & 11 & 9 & 14 \\
0 & 14 & 13 & 15 & 17 & 8 & 16 & 6 & 5 & 10 & 9 & 3 & 11 & 4 & 7 & 12 \\
0 & 15 & 14 & 13 & 16 & 17 & 7 & 4 & 3 & 8 & 12 & 6 & 9 & 5 & 10 & 11 \\
0 & 16 & 17 & 14 & 15 & 7 & 6 & 3 & 9 & 12 & 4 & 5 & 13 & 8 & 11 & 10 \\
17 & 0 & 1 & 11 & 14 & 3 & 4 & 2 & 12 & 13 & 7 & 8 & 5 & 10 & 6 & 9 \\
17 & 1 & 3 & 12 & 0 & 2 & 5 & 14 & 8 & 11 & 10 & 7 & 6 & 9 & 13 & 4 \\
17 & 3 & 2 & 1 & 8 & 4 & 14 & 7 & 0 & 9 & 5 & 11 & 10 & 6 & 12 & 13 \\
17 & 9 & 10 & 2 & 3 & 0 & 1 & 8 & 7 & 6 & 11 & 13 & 14 & 12 & 4 & 5 \\
17 & 10 & 9 & 3 & 2 & 1 & 8 & 0 & 11 & 7 & 13 & 12 & 4 & 14 & 5 & 6 \\
\hline
\end{array}
\]
References


