Abstract

Let $D$ be a set of positive integers. The \textit{kappa value} of $D$, denoted by $\kappa(D)$, is the parameter involved in the so-called "lonely runner conjecture." Let $x, y$ be positive integers, we investigate the kappa values for the family of sets $D = \{2, 3, x, y\}$. For a fixed positive integer $x > 3$, the exact values of $\kappa(D)$ are determined for $y = x + i$, $1 \leq i \leq 6$. These results lead to some asymptotic behavior of $\kappa(D)$ for $D = \{2, 3, x, y\}$.

1 Introduction

Let $D$ be a set of positive integers. For any real number $x$, let $||x||$ denote the minimum distance from $x$ to an integer, that is, $||x|| = \min\{\lceil x \rceil - x, x - \lfloor x \rfloor\}$. For any real $t$, denote $||tD||$ the smallest value $||td||$ among all $d \in D$. The \textit{kappa value} of $D$, denoted by $\kappa(D)$, is the supremum of $||tD||$ among all real $t$. That is,

$$\kappa(D) := \sup\{\alpha : ||tD|| \geq \alpha \text{ for some } t \in \mathbb{R}\}.$$ 

Wills [20] conjectured that $\kappa(D) \geq 1/(|D|+1)$ is true for all finite sets $D$. This conjecture is also known as the \textit{lonely runner conjecture} by Bienia et al. [2]. Suppose $m$ runners run laps on a circular track of unit circumference.

*Supported in part by the National Science Foundation under grant MS-1247679.
Each runner maintains a constant speed, and the speeds of all the runners are distinct. A runner is called *lonely* if the distance on the circular track between him or her and every other runner is at least $1/m$. Equivalently, the conjecture asserts that for each runner, there is some time $t$ when he or she becomes lonely. The conjecture has been proved true for $|D| \leq 6$ (cf. [1, 3, 6, 7]), and remains open for $|D| \geq 7$.

The parameter $\kappa(D)$ is closely related to another parameter of $D$ called the “density of integral sequences with missing differences.” For a set $D$ of positive integers, a sequence $S$ of non-negative integers is called a $D$-sequence if $|x - y| \not\in D$ for any $x, y \in S$. Denote $S(n)$ as $|S \cap \{0, 1, 2, \ldots, n - 1\}|$. The upper density $\overline{\delta}(S)$ and the lower density $\underline{\delta}(S)$ of $S$ are defined, respectively, by $\overline{\delta}(S) = \lim_{n \to \infty} S(n)/n$ and $\underline{\delta}(S) = \lim_{n \to \infty} S(n)/n$. We say $S$ has density $\delta(S)$ if $\overline{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The parameter of interest is the *density of $D$*, $\mu(D)$, defined by

$$\mu(D) := \sup \{ \delta(S) : S \text{ is a } D\text{-sequence} \}.$$  

It is known that for any set $D$ (cf. [4]):

$$\mu(D) \geq \kappa(D). \tag{1}$$

For two-element sets $D = \{a, b\}$, Cantor and Gordon [4] proved that $\kappa(D) = \mu(D) = \left\lfloor \frac{a+b}{a+b} \right\rfloor$. For 3-element sets $D$, if $D = \{a, b, a + b\}$ it was proved that $\kappa(D) = \mu(D)$ and the exact values were determined (see Theorem 2 below). For the general case $D = \{i, j, k\}$, various lower bounds of $\kappa(D)$ were given by Gupta [11], in which the values of $\mu(D)$ were also studied. In addition, among other results it was shown in [11] that if $D$ is an arithmetic sequence then $\kappa(D) = \mu(D)$ and the value was determined.

The parameters $\kappa(D)$ and $\mu(D)$ are closely related to coloring parameters of distance graphs. Let $D$ be a set of positive integers. The *distance graph generated by* $D$, denoted as $G(\mathbb{Z}, D)$, has all integers $\mathbb{Z}$ as the vertex set. Two vertices are adjacent whenever their absolute value difference falls in $D$. The *chromatic number* (minimum number of colors in a proper vertex-coloring) of the distance graph generated by $D$ is denoted by $\chi(D)$. It is known that $\chi(D) \leq \lceil 1/\kappa(D) \rceil$ for any set $D$ (cf. [21]).

The *fractional chromatic number* of a graph $G$, denoted by $\chi_f(G)$, is the minimum ratio $m/n$ ($m, n \in \mathbb{Z}^+$) of an $(m/n)$-coloring, where an $(m/n)$-coloring is a function on $V(G)$ to $n$-element subsets of $[m] = \{1, 2, \ldots, m\}$.
such that if $uv \in E(G)$ then $f(u) \cap f(v) = \emptyset$. It is known that for any graph $G$, $\chi_f(G) \leq \chi(G)$ (cf. [21]).

Denote the fractional chromatic number of $G(Z, D)$ by $\chi_f(D)$. Chang et al. [5] proved that for any set of positive integers $D$, it holds that $\chi_f(D) = 1/\mu(D)$. Together with (1) we obtain

$$\frac{1}{\mu(D)} = \chi_f(D) \leq \chi(D) \leq \lceil \frac{1}{\kappa(D)} \rceil. \tag{2}$$

The chromatic number of distance graphs $G(Z, D)$ with $D = \{2, 3, x, y\}$ was studied by several authors. For prime numbers $x$ and $y$, the values of $\chi(D)$ for this family were first studied by Eggleton, Erdős and Skilton [10] and later on completely solved by Voigt and Walther [18]. For general values of $x$ and $y$, Kemnitz and Kolberg [13] and Voigt and Walther [19] determined $\chi(D)$ for some values of $x$ and $y$. This problem was completely solved for all values of $x$ and $y$ by Liu and Setudja [15], in which $\kappa(D)$ was utilized as one of the main tools. In particular, it was proved in [15] that $\kappa(D) \geq 1/3$ for many sets in the form $D = \{2, 3, x, y\}$. By (2), for those sets it holds that $\chi(D) = 3$.

In this article we further investigate those previously established lower bounds of $\kappa(D)$ for the family of sets $D = \{2, 3, x, y\}$. In particular, we determine the exact values of $\kappa(D)$ for $D = \{2, 3, x, y\}$ with $|x - y| \leq 6$. Furthermore, for some cases it holds that $\kappa(D) = \mu(D)$. Our results also lead to asymptotic behavior of $\kappa(D)$.

2 Preliminaries

We introduce terminologies and known results that will be used to determine the exact values of $\kappa(D)$. It is easy to see that if the elements of $D$ have a common factor $r$, then $\kappa(D) = \kappa(D')$ and $\mu(D) = \mu(D')$, where $D' = D/r = \{d/r : d \in D\}$. Thus, throughout the article we assume that $\text{gcd}(D) = 1$, unless it is indicated otherwise.

The following proposition is derived directly from definitions.

Proposition 1. If $D \subseteq D'$ then $\kappa(D) \geq \kappa(D')$ and $\mu(D) \geq \mu(D')$.

The next result was established by Liu and Zhu [16], after confirming a conjecture of Rabinowitz and Proulx [17].
**Theorem 2.** [16] Suppose $M = \{a, b, a + b\}$ for some positive integers $a$ and $b$ with $\gcd(a, b) = 1$. Then

$$\mu(M) = \kappa(M) = \max \left\{ \frac{b}{\alpha} \cdot \frac{2a + b}{2b + a}, \frac{2b + a}{\alpha} \cdot \frac{2a + b}{2b + a} \right\}.$$ 

By Proposition 1, if $\{a, b, a + b\} \subseteq D$ for some $a$ and $b$, then Theorem 2 gives an upper bound for $\kappa(D)$.

For a $D$-sequence $S$, denote $S[n] = |\{0, 1, \ldots, n\} \cap S|$. The next result was proved by Haralambis [12].

**Lemma 3.** [12] Let $D$ be a set of positive integers, and let $\alpha \in (0, 1]$. If for every $D$-sequence $S$ with $0 \in S$ there exists a positive integer $n$ such that $\frac{S[n]}{n + 1} \leq \alpha$, then $\mu(D) \leq \alpha$.

For a given $D$-sequence $S$, we shall write elements of $S$ in an increasing order, $S = \{s_0, s_1, s_2, \ldots\}$ with $s_0 < s_1 < s_2 < \ldots$, and denote its difference sequence by

$$\Delta(S) = \{\delta_0, \delta_1, \delta_2, \ldots\}$$

where $\delta_i = s_{i+1} - s_i$.

We call a subsequence of consecutive terms in $\Delta(S)$, $\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1}$, generates a periodic interval of $k$ copies, $k \geq 1$, if $\delta_{j(a+b)+i} = \delta_{a+i}$ for all $0 \leq i \leq b-1$, $1 \leq j \leq k-1$. We denote such a periodic subsequence of $\Delta(S)$ by $(\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1})^k$. If the periodic interval repeats infinitely, then we simply denote it by $(\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1})$. If $\Delta(S)$ is infinite periodic, except the first finite number of terms, with the periodic interval $(t_1, t_2, \ldots, t_k)$, then the density of $S$ is $k/(\sum_{i=1}^{k} t_i)$.

**Proposition 4.** A sequence of non-negative integers $S$ is a $D$-sequence if and only if $\sum_{i=a}^{b} \delta_i \not\in D$ for every $a \leq b$.

**Proposition 5.** Assume $2, 3 \in D$. If $S$ is a $D$-sequence, then $\delta_i + \delta_{i+1} \geq 5$ for all $i$. The equality holds only when $\{\delta_i, \delta_{i+1}\} = \{1, 4\}$. Consequently, $\mu(D) \leq 2/5$.

**Lemma 6.** Let $D = \{2, 3\} \cup A$. Then $\kappa(D) = 2/5$ if and only if $A \subseteq \{x : x \equiv 2, 3 \mod 5\}$. Furthermore, if $\kappa(D) = 2/5$, then $\mu(D) = 2/5$. 

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Proof. Let $D = \{2, 3\} \cup A$. Assume $A \subseteq \{x : x \equiv 2, 3 \pmod{5}\}$. Let $t = 1/5$. Then $||td|| \geq 2/5$ for all $d \in D$. Hence $\kappa(D) \geq 2/5$. On the other hand, the density of the infinite periodic $D$-sequence $S$ with $\Delta(S) = (1, 4)$ is 2/5. By Proposition 5, this is an optimal $D$-sequence. Hence, $\mu(D) = 2/5$, implying $\kappa(D) = 2/5$.

Conversely, assume $\kappa(D) = 2/5$. Then $\mu(D) \geq 2/5$. By Proposition 5, $\mu(D) = 2/5$. By Proposition 4, this implies that if $d \in D$, then $d \not\equiv 0, 1, 4 \pmod{5}$. Thus the result follows.

Note, in $D = \{2, 3, x, y\}$, if $x = 1$, then it is known [16] and easy to see that $\mu(D) = \kappa(D) = 1/4$ if $y$ is not a multiple of 4 (with $\Delta(S) = (4)$); otherwise $y = 4k$ and $\mu(D) = \kappa(D) = k/(4k + 1)$ (with $\Delta(S) = ((4)^{k-1}5)$). Hence throughout the article we assume $x > 3$.

Another method we will utilize is an alternative definition of $\kappa(D)$. In this definition, for a projected lower bound $\alpha$ of $\kappa(D)$, for each element $z$ in $D$ the valid time $t$ for $z$ to achieve $\alpha$ is expressed as a union of disjoint intervals. Let $\alpha \in (0, 1/2)$. For positive integer $i$, define $I_i(\alpha) = \{t \in (0, 1) : \|ti\| \geq \alpha\}$. Equivalently,

$$I_i(\alpha) = \{t : n + \alpha \leq ti \leq n + 1 - \alpha, 0 \leq n \leq i - 1\}.$$

That is, $I_i$ consists of intervals of reals with length $(1 - 2\alpha)/i$ and centered at $(2n + 1)/(2i)$, $n = 0, 1, \ldots, i - 1$. By definition, $\kappa(D) \geq \alpha$ if and only if $\bigcap_{i \in D} I_i(\alpha) \neq \emptyset$. Thus,

$$\kappa(D) = \sup \left\{ \alpha \in (0, \frac{1}{2}) : \bigcap_{i \in D} I_i(\alpha) \neq \emptyset \right\}.$$

Observe that if $\bigcap_{i \in D} I_i(\alpha)$ consists of only isolated points, then $\kappa(D) \leq \alpha$. Hence, we have the following:

**Proposition 7.** For a set $D$, $\kappa(D) \leq d/c$ if $\bigcap_{i \in D} I_i$ is a set of isolated points, where

$$I_i = \bigcup_{n=0}^{i-1} \left[ \frac{d + cn}{i}, \frac{c - d + cn}{i} \right].$$
3 \quad D = \{2, 3, x, y\} \text{ for } y = x + 1, x + 2, x + 3

Theorem 8. Let \( D = \{2, 3, x, x + 1\}, \, x \geq 4 \). Then

\[ \kappa(D) = \mu(D) = \begin{cases} \frac{2(x+3)+1}{x+3} & \text{if } x \equiv 1 \pmod{5}; \\ \frac{2(x+3)}{x+3} & \text{otherwise}. \end{cases} \]

Proof. We prove the following cases.

Case 1. \( x = 5k + 2 \). The result follows by Lemma 6.

Case 2. \( x = 5k + 3 \). Let \( t = (k+1)/(5k+6) \). Then \( ||dt|| \geq (2k+2)/(5k+6) \) for every \( d \in D \). Hence \( \kappa(D) \geq (2k+2)/(5k+6) \).

By (1) it remains to show that \( \mu(D) \leq (2k+2)/(5k+6) \). Assume to the contrary that \( \mu(D) > (2k+2)/(5k+6) \). By Lemma 3, there exists a \( D \)-sequence \( S \) with \( S[n]/(n+1) > (2k+2)/(5k+6) \) for all \( n \geq 0 \). This implies, for instance, \( S[0] \geq 1 \), so \( s_0 = 0 \); \( S[2] \geq 2 \), so \( s_1 = 1 \) (as \( 2, 3 \in D \)); \( S[5] \geq 3 \), so \( s_3 = 5 \). Moreover, \( S[5k+5] \geq 2k+3 \). By Proposition 5, it must be \( (\delta_0, \delta_1, \delta_2, \ldots, \delta_{2k+1}) = (1, 4, 1, 4, \ldots, 1, 4) \). This implies \( 5k+5 \in S \), which is impossible since \( 1 \in S \) and \( 5k+4 \notin D \). Therefore, \( \mu(D) = \kappa(D) = (2k+2)/(5k+6) \).

Case 3. \( x = 5k + 4 \). Let \( t = (k+1)/(5k+7) \). Then \( ||dt|| \geq (2k+2)/(5k+7) \) for all \( d \in D \). Hence \( \kappa(D) \geq (2k+2)/(5k+7) \).

By (1) it remains to show that \( \mu(D) \leq (2k+2)/(5k+7) \). Assume to the contrary that \( \mu(D) > (2k+2)/(5k+7) \). By Lemma 3, there exists a \( D \)-sequence \( S \) with \( S[n]/(n+1) > (2k+2)/(5k+7) \) for all \( n \geq 0 \). This implies, for instance, \( S[0] \geq 1 \), so \( s_0 = 0 \); \( S[3] \geq 2 \), so \( s_1 = 1 \) (as \( 2, 3 \in D \)); and \( S[5k+6] \geq 2k+3 \). By Proposition 5, either \( 5k+5 \) or \( 5k+6 \) is an element in \( S \). This is impossible since \( 0, 1 \in S \) and \( 5k+4, 5k+5 \in D \). Thus \( \mu(D) = \kappa(D) = (2k+2)/(5k+7) \).

Case 4. \( x = 5k + 5 \). Let \( t = (k+1)/(5k+8) \). Then \( ||dt|| \geq (2k+2)/(5k+8) \) for all \( d \in D \). Hence \( \kappa(D) \geq (2k+2)/(5k+8) \).

It remains to show \( \mu(D) \leq (2k+2)/(5k+8) \). Assume to the contrary that \( \mu(D) > (2k+2)/(5k+8) \). By Lemma 3, there exists a \( D \)-sequence \( S \) with \( S[n]/(n+1) > (2k+2)/(5k+8) \) for all \( n \geq 0 \). Similar to the above, one has \( 0, 1 \in S \) and \( S[5k+7] \geq 2k+3 \). This implies that one of \( 5k+5 \), \( 5k+6 \), or \( 5k+7 \) is an element in \( S \), which is again impossible. Therefore, \( \mu(D) = \kappa(D) = (2k+2)/(5k+8) \).
Case 5. $x = 5k + 1$. Let $t = (k+1)/(5k+4)$. Then $|dt| \geq (2k+1)/(5k+4)$ for all $d \in D$. Hence $\kappa(D) \geq (2k+1)/(5k+4)$.

Now we show $\mu(D) \leq (2k + 1)/(5k + 4)$. Assume to the contrary that $\mu(D) > (2k + 1)/(5k + 4)$. By Lemma 3, $(s_0, s_1) = (0, 1)$, and $S[5k + 3] \geq 2k + 2$. Because $S[5k] \leq 2k + 1$, so $S \cap \{5k + 1, 5k + 2, 5k + 3\} \neq \emptyset$, which is impossible. Therefore, $\mu(D) = \kappa(D) = (2k + 1)/(5k + 4)$. □ □

By the above proofs, one can extend the family of sets $D$ to the following:

Corollary 9. Let $D = \{2, 3, x, x + 1\} \cup D'$, where $D' \subseteq \{y : y \equiv \pm 2, \pm 3 \ (\text{mod} \ (x + 3))\}$. Then $\mu(D) = \kappa(D) = \mu(\{2, 3, x, x + 1\})$.

Corollary 10. Let $D = \{2, 3, x, x + 1\}$. Then

$$\lim_{x \to \infty} \kappa(D) = \frac{2}{5}.$$ 

Theorem 11. Let $D = \{2, 3, x, x + 2\}$, $x \geq 4$. Assume $x + 4 = 6\beta + r$ with $0 \leq r \leq 5$. Then

$$\kappa(D) = \begin{cases} \left\lfloor \frac{x+4}{x+1} \right\rfloor & \text{if } 0 \leq r \leq 2; \\ \left\lfloor \frac{x+4}{2x+2} \right\rfloor & \text{if } 3 \leq r \leq 5. \end{cases}$$

Furthermore, $\kappa(D) = \mu(D)$ if $r \neq 3$.

Proof. We prove the following cases.

Case 1. $x = 6k + 2$. Then $r = 0$. Let $t = 1/6$. Then $|dt| \geq 1/3$ for all $d \in D$. Hence $\kappa(D) \geq 1/3$.

Now we prove $\mu(D) \leq 1/3$. Let $M' = \{2, x, x + 2\} = \{2, 6k + 2, 6k + 4\}$. By Theorem 2 with $M = \{1, 3k+1, 3k+2\}$, we obtain $\mu(M') = \mu(M) = 1/3$. Because $M' \subseteq D$, so $\mu(D) \leq \mu(M') = 1/3$.

Case 2. $x = 6k + 3$. Then $r = 1$. Let $t = (k + 1)/(6k + 7)$. Then $|dt| \geq (2k + 2)/(6k + 7)$ for all $d \in D$. Hence $\kappa(D) \geq (2k + 2)/(6k + 7)$.

By Theorem 2 with $M = \{2, x, x + 2\} = \{2, 6k + 3, 6k + 5\}$, we get $\mu(M) = (2k + 2)/(6k + 7)$. Because $M \subseteq D$, so $\mu(D) \leq \mu(M) = (2k + 2)/(6k + 7)$. Thus, the result follows.

Case 3. $x = 6k + 4$. Then $r = 2$. Let $t = (k + 1)/(6k + 8)$. Then $|dt| \geq (2k + 2)/(6k + 8)$ for all $d \in D$. Hence $\kappa(D) \geq (2k + 2)/(6k + 8)$.
By Theorem 2 with $M = \{2, x, x + 2\} = \{2, 6k + 4, 6k + 6\}$ which can be reduced to $M' = \{1, 3k + 2, 3k + 3\}$, we obtain $\mu(M) = (k + 1)/(3k + 4)$. Therefore, $\mu(D) \leq \mu(M) = (2k + 2)/(6k + 8)$. So the result follows.

**Case 4.** $x = 6k + 5$. Then $r = 3$. Let $t = (2k + 3)/(12k + 12)$. Then $||dt|| \geq (4k + 3)/(12k + 12)$ for all $d \in D$. Hence $\kappa(D) \geq (4k + 3)/(12k + 12)$.

By Proposition 7, it remains to show that $\bigcap_{i=2,3,x,x+2} I_i$ is a set of isolated points, where

$$I_i = \bigcup_{n=0}^{i-1} \left[ \frac{4k + 3 + n(12k + 12)}{i}, \frac{8k + 9 + n(12k + 12)}{i} \right].$$

Let $I = \bigcap_{i=2,3,x,x+2} I_i$. By symmetry it is enough to consider the interval $I \cap [0, (12k + 12)/2]$. In the following we claim $I \cap [0, 6k + 6] = \{2k + 3\}$. (Indeed, this single point is the numerator of the $t$ value at the beginning of the proof.)

Note that $I_2 \cap I_3 \cap [0, 6k + 6] = [(4k + 3)/2, (8k + 9)/3]$. Denote this interval by

$$I_{2,3} = \left[ \frac{4k + 3}{2}, \frac{8k + 9}{3} \right].$$

We then begin to investigate possible values of $n$ for $I_x$ and $I_{x+2}$, respectively, that will fall within $I_{2,3}$. First, we compare the $I_x$ intervals with $I_{2,3}$. Recall

$$I_x = \left[ \frac{3 + 4k + n(12 + 12k)}{6k + 5}, \frac{8k + 9 + n(12 + 12k)}{6k + 5} \right], \quad 0 \leq n \leq 6k + 4.$$

By calculation, the intervals of $I_x$ that intersect with $I_{2,3}$ are those with $n \geq k$. Similarly, we compare $I_{x+2}$ intervals with $I_{2,3}$. Recall

$$I_{x+2} = \left[ \frac{3 + 4k + n(12 + 12k)}{6k + 7}, \frac{8k + 9 + n(12 + 12k)}{6k + 7} \right], \quad 0 \leq n \leq 6k + 6.$$

By calculation, the intervals of $I_{x+2}$ that intersect with $I_{2,3}$ are those with $n \geq k + 1$.

Next, we consider the intersection between intervals of $I_x$ and $I_{x+2}$. Let $n = k + a$ for some $a \geq 0$ for the $I_x$ interval, and let $n = k + a'$ for some $a' \geq 1$ for the $I_{x+2}$ interval. By taking the common denominator of the $I_x$ and $I_{x+2}$ intervals we obtain the following numerators of those intervals:

for $I_x : [21 + 84a + 130k + 156ak + 180k^2 + 72ak^2 + 72k^3]$, 

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63 + 84a + 194k + 156ak + 204k^2 + 72ak^2 + 72k^3];
for \( I_{x+2} : [15 + 60a' + 98k + 132a'k + 156k^2 + 72a'k^2 + 72k^3, \)
\[45 + 60a' + 154k + 132a'k + 180k^2 + 72a'k^2 + 72k^3].\]

Using \( a = a' = 1 \), we get
\[\text{for } I_x : [105 + 286k + 252k^2 + 72k^3, 147 + 350k + 276k^2 + 72k^3]\]
\[\text{for } I_{x+2} : [75 + 230k + 228k^2 + 72k^3, 105 + 286k + 252k^2 + 72k^3].\]

Thus, there is a single point intersection for \( I_x \) and \( I_{x+2} \) when \( a = a' = 1 \), which is \( \{2k+3\} \). This single point intersection is also within the \( I_{2,3} \) interval. Hence, \( \{2k + 3\} \in I \cap [0, 6k + 6] \).

In addition, through inspection it is clear that making \( n = k \) (i.e. \( a = 0 \)) for the \( I_x \) interval and \( n \geq k+1 \) (\( a' \geq 1 \)) for the \( I_{x+2} \) interval removes \( I_x \) and \( I_{x+2} \) from intersecting one another. For all other cases, \( a = 1 \) and \( a' \geq 2, a \geq 2 \) and \( a' = 1, \) or \( a, a' \geq 2, \) there will never be an intersection of intervals for all elements in \( D, \) either because the \( I_{2,3} \) interval is too small or because the \( I_{x+2} \) elements become too big. Thus, \( I \cap [0, 6k + 6] = \{2k + 3\} \).

**Case 5.** \( x = 6k + 6. \) Then \( r = 4. \) Let \( t = (2k + 3)/(12k + 14). \) Then \( ||dt|| \geq (4k + 4)/(12k + 14) \) for all \( d \in D. \) Hence \( \kappa(D) \geq (4k + 4)/(12k + 14) \).

By Theorem 2 with \( M = \{2, x, x+2\} = \{2, 6k + 6, 6k + 8\} \) which can be reduced to \( M' = \{1, 3k + 3, 3k + 4\} \), we get \( \mu(M) = \kappa(M) = (2k + 2)/(6k + 7). \) Hence, \( \mu(D) \leq \mu(M) = (2k + 2)/(6k + 7). \)

**Case 6.** \( x = 6k + 7. \) Then \( r = 5. \) Let \( t = (2k + 3)/(12k + 16). \) Then \( ||dt|| \geq (4k + 5)/(12k + 16) \) for all \( d \in D. \) Hence \( \kappa(D) \geq (4k + 5)/(12k + 16) \).

By Theorem 2 with \( M = \{2, x, x + 2\} = \{2, 6k + 7, 6k + 9\} \), we obtain \( \mu(M) = \kappa(M) = (4k + 5)/(12k + 16). \) Therefore, \( \mu(D) \leq (4k + 5)/(12k + 16). \)

**Theorem 12.** Let \( D = \{2, 3, x, x + 3\}, x \geq 4. \) Assume \( 2x + 3 = 9\beta + r \) with \( 0 \leq r \leq 8. \) Then
\[
\kappa(D) = \begin{cases} 
\frac{3(2x+3)}{2x+3} & \text{if } 0 \leq r \leq 5; \\
\frac{r+5}{r+6} & \text{if } 6 \leq r \leq 8. 
\end{cases}
\]

Furthermore, if \( r = 0, 1, 3, 6, 8 \) then \( \kappa(D) = \mu(D). \)
Theorem 11. Let

\[ \text{Case 5.} \]

Then \( r = 0 \). Let \( t = 2/9 \). Then \( ||dt|| \geq 1/3 \) for all \( d \in D \). Hence \( \kappa(D) \geq (6k + 3)/(18k + 9) = 1/3 \).

By Theorem 2 with \( M = \{3, x, x + 3\} = \{3, 9k + 3, 9k + 6\} \), which can be reduced to \( M' = \{1, 3k + 1, 3k + 2\} \), resulting in \( \mu(M) = \kappa(M) = 1/3 \). Because \( M \subseteq D \), so \( \mu(D) = \mu(M) = 1/3 \).

\[ \text{Case 2.} x = 9k + 8. \] Then \( r = 1 \). Let \( t = (4k + 4)/(18k + 19) \). Then \( ||dt|| \geq (6k + 6)/(18k + 19) \) for all \( d \in D \). Hence \( \kappa(D) \geq (6k + 6)/(18k + 19) \).

By Theorem 2 with \( M = \{3, x, x + 3\} \), we get \( \kappa(M) = (6k + 6)/(18k + 19) \). Hence, \( \mu(D) \leq \kappa(M) = (6k + 6)/(18k + 19) \).

\[ \text{Case 3.} x = 9k + 4. \] Then \( r = 2 \). Let \( t = (4k + 2)/(18k + 11) \). Then \( ||dt|| \geq (6k + 3)/(18k + 11) \) for all \( d \in D \). Thus, \( \kappa(D) \geq (6k + 3)/(18k + 11) \).

The proof for the other direction is similar to the proof of Case 4 in Theorem 11. Let \( I = \bigcap_{i=2,3,x,x+3} I_i \). By calculation we have \( I \cap [0, 9k+(11/2)] = \{4k + 2\} \). This single point of intersection occurs when \( n = 2k \) in the \( I_x \) interval, and \( n = 2k + 1 \) in the \( I_{x+3} \) interval.

\[ \text{Case 4.} x = 9k. \] Then \( r = 3 \). Let \( t = 4k/(18k + 3) \). Then \( ||dt|| \geq (6k)/(18k + 3) \) for all \( d \in D \). Thus \( \kappa(D) \geq (2k)/(6k + 1) \).

By Theorem 2 with \( M = \{3, x, x + 3\} = \{3, 9k, 9k + 3\} \), \( \mu(M) = \kappa(M) = (2k)/(6k + 1) \). Hence, the result follows.

\[ \text{Case 5.} x = 9k + 5. \] Then \( r = 4 \). Let \( t = (4k + 2)/(18k + 13) \). Then \( ||dt|| \geq (6k + 3)/(18k + 13) \) for all \( d \in D \). Thus \( \kappa(D) \geq (6k + 3)/(18k + 13) \).

The proof for the other direction is similar to the proof of Case 4 in Theorem 11. Let \( I = \bigcap_{i=2,3,x,x+3} I_i \). By calculation we have \( I \cap [0, 9k+(13/2)] = \{4k + 2\} \). This single point of intersection occurs when \( n = 2k \) in the \( I_x \) interval, and \( n = 2k + 1 \) in the \( I_{x+3} \) interval.

\[ \text{Case 6.} x = 9k + 1. \] Then \( r = 5 \). Let \( t = (4k)/(18k + 5) \). Then \( ||dt|| \geq (6k)/(18k + 5) \) for all \( d \in D \). Thus \( \kappa(D) \geq (6k)/(18k + 5) \).

The proof for \( \kappa(D) \leq (6k)/(18k + 5) \) is similar to the proof of Case 4 in Theorem 11. Let \( I = \bigcap_{i=2,3,x,x+3} I_i \). By calculation we have \( I \cap [0, 9k+(5/2)] = \{4k\} \). This single point of intersection occurs when \( n = 2k - 1 \) in the \( I_x \) interval, and \( n = 2k \) in the \( I_{x+3} \) interval.

\[ \text{Case 7.} x = 9k + 6. \] Then \( r = 6 \). Let \( t = (2k + 3)/(9k + 12) \). Then \( ||dt|| \geq (3k + 3)/(9k + 12) \) for all \( d \in D \). Thus \( \kappa(D) \geq (k + 1)/(3k + 4) \).
By Theorem 2 with \( M = \{3, x, x + 3\} \) with \( M = \{3, x, x + 3\} = \{3, 9t + 6, 9t + 9\} \), which can be reduced to \( M' = \{1, 3t + 2, 3t + 3\} \), we get \( \mu(M) = \kappa(M) = (k + 1)/(3k + 4) \). Because \( M \subseteq D \), so \( \kappa(D) \leq \mu(D) \leq \mu(M) \leq \kappa(M) = (k + 1)/(3k + 4) \).

**Case 8.** \( x = 9k + 11 \). Then \( r = 7 \). Let \( t = (2k + 4)/(9k + 17) \). Then \( ||dt|| \geq (3k + 5)/(9k + 17) \) for all \( d \in D \). Thus \( \kappa(D) \geq (3k + 5)/(9k + 17) \).

The proof for the other direction is similar to the proof of Case 4 in Theorem 11. Let \( I = \bigcap_{i=2,3,x,x+3} I_i \). By calculation we have \( I \cap [0, 9k+(17/2)] = \{2k + 4\} \). This single point of intersection occurs when \( n = 2k + 2 \) in the \( I_x \) interval, and \( n = 2k + 3 \) in the \( I_{x+3} \) interval.

**Case 9.** \( x = 9k + 7 \). Then \( r = 8 \). Let \( t = (2k + 3)/(9k + 13) \). Then \( ||dt|| \geq (3k + 4)/(9k + 13) \) for all \( d \in D \). Thus \( \kappa(D) \geq (3k + 4)/(9k + 13) \).

By Theorem 2 with \( M = \{3, x, x + 3\} = \{3, 9t + 7, 9t + 10\} \), we get \( \kappa(M) = (3k + 4)/(9k + 13) \). Because \( M \subseteq D \), so \( \kappa(D) \leq \mu(D) \leq \mu(M) = \kappa(M) = (3k + 4)/(9k + 13) \).

**Corollary 13.** Let \( D = \{2, 3, x, y\} \) where \( y \in \{x + 2, x + 3\} \). Then

\[
\lim_{x \to \infty} \kappa(D) = \frac{1}{3}.
\]

**4.** \( D = \{2, 3, x, y\} \) for \( y = x + 4, x + 5, x + 6 \)

By similar proofs to the previous section, we obtain the following results.

**Theorem 14.** Let \( D = \{2, 3, x, x + 4\} \), \( x \geq 4 \). Assume \( (x + 4) = 5\beta + r \) with \( 0 \leq r \leq 4 \). Then

\[
\kappa(D) = \begin{cases} 
\frac{2\beta+r}{x+7} & \text{if } 0 \leq r \leq 1; \\
\mu(D) = \frac{2}{5} & \text{if } r = 2; \\
\frac{2\beta}{x+2} & \text{if } 3 \leq r \leq 4.
\end{cases}
\]

**Proof.** The case for \( r = 2 \) is from Lemma 6. The following table gives the corresponding \( t, \kappa(D) \), and the \( n \) values of \( I_x \) and \( I_{x+4} \) where the single intersection point occurs.
Theorem 15. Let $D = \{2, 3, x, x + 5\}$, $x \geq 4$. Assume $(x + 3) = 5\beta + r$ with $0 \leq r \leq 4$. Then

$$\kappa(D) = \begin{cases} 
\mu(D) = \frac{2}{5} & \text{if } 0 \leq r \leq 1; \\
\frac{2\beta}{x+2} & \text{if } 2 \leq r \leq 3; \\
\frac{2\beta+1}{x+3} & \text{if } r = 4.
\end{cases}$$

Proof. The cases for $r = 0, 1$ are by Lemma 6. The following table gives the corresponding $t, \kappa(D)$, and the $n$ values of $I_x$ and $I_{x+5}$ where the single intersection point occurs.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$r$</th>
<th>$t$</th>
<th>$n$ in $I_x$</th>
<th>$n$ in $I_{x+4}$</th>
<th>$\kappa(D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5k+4$</td>
<td>3</td>
<td>$(k+1)/(5k+6)$</td>
<td>$k$</td>
<td>$k+1$</td>
<td>$(2k+2)/(5k+6)$</td>
</tr>
<tr>
<td>$5k+5$</td>
<td>4</td>
<td>$(k+1)/(5k+7)$</td>
<td>$k$</td>
<td>$k+1$</td>
<td>$(2k+2)/(5k+7)$</td>
</tr>
<tr>
<td>$5k+6$</td>
<td>0</td>
<td>$(k+3)/(5k+13)$</td>
<td>$k+1$</td>
<td>$k+2$</td>
<td>$(2k+4)/(5k+13)$</td>
</tr>
<tr>
<td>$5k+7$</td>
<td>1</td>
<td>$(k+3)/(5k+14)$</td>
<td>$k+1$</td>
<td>$k+2$</td>
<td>$(2k+5)/(5k+14)$</td>
</tr>
<tr>
<td>$5k+8$</td>
<td>2</td>
<td>$1/5$</td>
<td></td>
<td></td>
<td>$2/5$</td>
</tr>
</tbody>
</table>

Theorem 16. Let $D = \{2, 3, x, x + 6\}$, $x \geq 4$. Assume $(x + 8) = 5\beta + r$ with $0 \leq r \leq 4$. Then

$$\kappa(D) = \begin{cases} 
\mu(D) = \frac{2}{7} & \text{if } x = 5; \\
\mu(D) = \frac{2}{5} & \text{if } r = 0; \\
\frac{2\beta}{x+8} & \text{if } 1 \leq r \leq 3 \text{ and } x \neq 5; \\
\frac{2\beta+1}{x+3} & \text{if } r = 4.
\end{cases}$$
Proof. Assume $x = 5$. That is $D = \{2, 3, 5, 11\}$. Letting $t = 1/7$ we get $\|td\| \geq 2/7$ for every $d \in D$. Hence, $\kappa(D) \geq 2/7$. On the other hand, by Theorem 2, $\mu(\{2, 3, 5\}) = 2/7$. Therefore, by (2), we have $\kappa(D) \leq \mu(D) \leq 2/7$.

The case for $r = 0$ is from Lemma 6. The following table gives the corresponding $t$, $\kappa(D)$, and the $n$ values of $I_x$ and $I_{x+6}$ where the single intersection point occurs.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$r$</th>
<th>$t$</th>
<th>$n$ in $I_x$</th>
<th>$n$ in $I_{x+6}$</th>
<th>$\kappa(D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5k + 4$</td>
<td>2</td>
<td>$(k + 2)/(5k + 12)$</td>
<td>$k$</td>
<td>$k + 1$</td>
<td>$(2k + 4)/(5k + 12)$</td>
</tr>
<tr>
<td>$5k + 5$</td>
<td>3</td>
<td>$(k + 2)/(5k + 13)$</td>
<td>$k$</td>
<td>$k + 1$</td>
<td>$(2k + 4)/(5k + 13)$</td>
</tr>
<tr>
<td>$5k + 6$</td>
<td>4</td>
<td>$(k + 2)/(5k + 9)$</td>
<td>$k + 1$</td>
<td>$k + 2$</td>
<td>$(2k + 3)/(5k + 9)$</td>
</tr>
<tr>
<td>$5k + 7$</td>
<td>0</td>
<td>$1/5$</td>
<td></td>
<td></td>
<td>$2/5$</td>
</tr>
<tr>
<td>$5k + 8$</td>
<td>1</td>
<td>$(k + 3)/(5k + 16)$</td>
<td>$k + 1$</td>
<td>$k + 2$</td>
<td>$(2k + 6)/(5k + 16)$</td>
</tr>
</tbody>
</table>

Corollary 17. Let $D = \{2, 3, x, y\}$ where $y \in \{x + 4, x + 5, x + 6\}$. Then

$$\lim_{x \to \infty} \kappa(D) = \frac{2}{5}.$$  

Concluding remark and future study. Similar to Corollary 9, one can obtain sets $D'$ that are extensions of the sets $D$ studied in this article, $D \subset D'$, such that $\kappa(D) = \kappa(D')$. In addition, the methods used in this article can be applied to other sets $D = \{2, 3, x, x + c\}$ with $c \geq 7$. For a fixed $c$, preliminary results we obtained thus far indicate that the values of $\kappa(D)$ might be inconsistent for the first finite terms, while after a certain threshold, they seem to be more consistent (that is, most likely it can be described by a single formula). Thus, we would like to investigate whether the conclusion of Corollary 17 holds for all $D = \{2, 3, x, y\}$, $x < y$, where $y \neq x + 2, x + 3$? In a broader sense, it is interesting to further study the asymptotic behavior of $\kappa(D)$ for sets $D$ containing 2 and 3, and identify any dominating factors for such behavior.

References


