THE ISOPERIMETRIC NUMBER OF A GENERALIZED PALEY GRAPH

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Abstract. Let $p$ be an odd prime, $m \geq 2$ be an integer, and $d = \gcd(m, p-1)$. Suppose that $d$ divides $(p-1)/2$. We define the generalized Paley graph on $p$ and $m$ to be the Cayley graph whose vertex set is $\mathbb{Z}_p$ and whose generating set is the set of non-zero $m$-th powers modulo $p$. We derive basic properties of these graphs. We give bounds on the isoperimetric number of a generalized Paley graph.

1. Introduction. Throughout this paper, let $\mathbb{Z}_p$ denote the integers modulo a prime $p$ and $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus \{0\}$.

We begin by recalling the definition of an undirected Cayley graph. Let $G$ be a group. Let $\Gamma$ be a symmetric subset of $G$, that is, $\gamma \in \Gamma$ if and only if $\gamma^{-1} \in \Gamma$. The Cayley graph of $G$ and $\Gamma$, denoted by $\text{Cay}(G, \Gamma)$, is defined to be the graph whose vertex set is $G$ and where two vertices $x$ and $y$ are adjacent iff $y^{-1}x \in \Gamma$.

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We now define generalized Paley graphs. We generalize the definition from [1] which only considers odd $m$. Let $p$ be an odd prime, $m \geq 2$ be an integer, and $d = \gcd(p - 1, m)$. Suppose that $d$ divides $(p - 1)/2$. Consider the set
\[ \Gamma^m_p = \{ a^m \mid a \in \mathbb{Z}_p^X \} \]
of $m$-th powers of non-zero integers modulo $p$. We define the generalized Paley graph $X^m_p$ to be the Cayley graph $\text{Cay}(\mathbb{Z}_p, \Gamma^m_p)$.

We will see in Proposition 3 that the condition that $d$ divides $(p - 1)/2$ ensures that the generating set for the generalized Paley graph is symmetric which gives us an undirected graph. One may also consider directed graphs by relaxing this restriction, but we do not do that in this paper.

For example, let $p = 17$ and $m = 6$. Then $d = 2$ which divides $(p - 1)/2 = 8$. We have that $\Gamma^6_{17} = \{1, 2, 4, 8, 9, 13, 15, 16\}$. See Figure 1 for a picture of $X^6_{17}$.

When $m = 2$ and $p \equiv 1(\text{mod } 4)$ we get the standard definition of a Paley graph.

Let $X$ be a graph with vertex set $V$. Given a subset of vertices $F$, the boundary of $F$, denoted by $\partial F$, is defined to be the set of edges of $X$ with one endpoint in $F$ and one endpoint in $V \setminus F$. The isoperimetric number, or Cheeger constant, of $X$ is defined to be
\[ h(X) = \min \left\{ \frac{|\partial F|}{|F|} \mid F \subseteq V \text{ and } 0 < |F| \leq |V|/2 \right\}. \]
In general it is a difficult combinatorial problem to get an exact value for the isoperimetric number of a graph. Instead one gives approximations, which is what we do in this paper. In particular, we generalize the following result from [4]. Let $p \equiv 1 \pmod{4}$ and $m = 2$. Then as mentioned above we get that $X_p^2$ is a standard Paley graph. It was shown in [4] that

$$\frac{p - \sqrt{p}}{4} \leq h(X_p^2) \leq \frac{p - 1}{4}. \quad (1)$$

This implies that $\lim_{p \to \infty} h(X_p^2)/p = 1/4$, where the limit is over primes congruent to 1 modulo 4.

In this paper, we generalize equation (1) to $m \geq 2$. In particular, we derive the following proposition.

**Proposition 1.** Let $p$ be an odd prime, $m \geq 2$ be an integer, and $d = \gcd(p - 1, m)$. Assume that $d$ divides $(p - 1)/2$ and that $d > 1$. Then

$$\frac{p + (1 - d)\sqrt{p}}{2d} \leq h(X_p^m) \leq \frac{(d - 1)p + M_m\sqrt{p} + (2d - 1)}{d^2}. \quad (2)$$

Here $M_m$ is the number of triples of rational numbers $(\alpha_0, \alpha_1, \alpha_2)$ satisfying $m\alpha_0, m\alpha_1, m\alpha_2 \in \mathbb{Z}$, $\alpha_0 + \alpha_1 + \alpha_2 \in \mathbb{Z}$, and $0 < \alpha_0, \alpha_1, \alpha_2 < 1$.

Note that when $d = 1$ we have that $X_p^m$ is a complete graph (see Proposition 5) whose isoperimetric number is well-known.

See Sections 3 and 4 for a proof of Proposition 1. The lower bound of Proposition 1 is derived using an estimate on the eigenvalues of $X_p^m$ (which are essentially Gauss sums). The upper bound is derived by estimating the number of solutions to the equation $x^m + y^m + z^m = 0$ modulo $p$.

When $m = 2$ we get that Proposition 1 gives essentially the same result as equation (1). Suppose that $p \equiv 1 \pmod{4}$ and $m = 2$. Then $d = \gcd(p - 1, 2) = 2$. The integer $M_2$ is the number of triplets of rational numbers $(\alpha_0, \alpha_1, \alpha_2)$ satisfying the following conditions: $2\alpha_0, 2\alpha_1, 2\alpha_2 \in \mathbb{Z}$, $\alpha_0 + \alpha_1 + \alpha_2 \in \mathbb{Z}$ and $0 < \alpha_0, \alpha_1, \alpha_2 < 1$. This implies that $M_2 = 0$. In this case, Proposition 1 becomes

$$\frac{p - \sqrt{p}}{4} \leq h(X_p^2) \leq \frac{p + 3}{4}, \quad (2)$$

which is asymptotically equivalent to equation (1).

We now consider the case when $m = 3$. Suppose that $p$ is an odd prime. If $p \equiv 0 \pmod{3}$ or $p \equiv 2 \pmod{3}$, then $d = \gcd(p - 1, 3) = 1$. This does not
Table 1. Approximate values where $m = 3$ and $p \equiv 1 \pmod{3}$

<table>
<thead>
<tr>
<th>$p$</th>
<th>lower bound from Prop (1)</th>
<th>$h(X_p^3)$</th>
<th>$\gamma$-bound from Prop (2)</th>
<th>upper bound from Prop (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.28475</td>
<td>0.6667</td>
<td>0.66667</td>
<td>2.69906</td>
</tr>
<tr>
<td>13</td>
<td>0.964816</td>
<td>1.6667</td>
<td>2</td>
<td>4.24568</td>
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<td>?</td>
<td>3.55556</td>
<td>5.74642</td>
</tr>
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<td>17,349.3</td>
<td>?</td>
<td>17,465.8</td>
<td>23,348.7</td>
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<tr>
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<td>?</td>
<td>17,529.3</td>
<td>23,367.4</td>
</tr>
<tr>
<td>1,299,709</td>
<td>216,238</td>
<td>?</td>
<td>216,659</td>
<td>289,078</td>
</tr>
<tr>
<td>2,750,161</td>
<td>457,807</td>
<td>?</td>
<td>458,707</td>
<td>611,516</td>
</tr>
</tbody>
</table>

satisfy Proposition 1. Indeed, we will see in Proposition 5 that when $d = 1$ we get a complete graph, whose isoperimetric number is known. Now consider the infinite family of generalized Paley graphs $X_p^3$ where $p \equiv 1 \pmod{3}$. In this case $d = \gcd(p - 1, 3) = 3$ which divides $(p - 1)/2$. The integer $M_3$ is the number of triplets of rational numbers $(\alpha_0, \alpha_1, \alpha_2)$ satisfying the following conditions: $3\alpha_0, 3\alpha_1, 3\alpha_2 \in \mathbb{Z}$, $\alpha_0 + \alpha_1 + \alpha_2 \in \mathbb{Z}$ and $0 < \alpha_0, \alpha_1, \alpha_2 < 1$. This implies that $M_3 = 2$. Proposition 1 gives that

$$\frac{p - 2\sqrt{p}}{6} \leq h(X_p^3) \leq \frac{2p + 2\sqrt{p} + 5}{9}$$

See Table 1 for some example calculations of equation (3).

One can try other values of $m$. For example, any odd power $m$ and $p \equiv 1 \pmod{m}$ give $d = m$ in equation (1).

We derive a second upper bound for $h(X_p^m)$, which we call the $\gamma$-bound. This result is a generalization of the $\alpha$-bound given in [4] for regular Paley graphs ($m = 2$). The $\gamma$-bound appears to give a better upper bound than Proposition 1 does. However, it is harder to deal with and is not in closed form.

In Proposition 3 it is shown that $\Gamma_p^m$ is a symmetric subset of $\mathbb{Z}_p$ iff $d|(p - 1)/2$, and therefore we can write it as

$$\Gamma_p^m = \{\gamma_1, \gamma_2, \cdots, \gamma_k, -\gamma_k, \cdots, -\gamma_2, -\gamma_1\}$$

where $k = \frac{p - 1}{2d}$ and $0 \leq \gamma_i \leq (p - 1)/2$. The proof of the following Proposition is given in Section 5.

**Proposition 2** (The $\gamma$-bound). The isoperimetric number of a generalized Paley graph satisfies the bound

$$h(X_p^m) \leq \frac{4}{p - 1} \sum_{i=1}^{k} \gamma_i$$
where $\Gamma_p^m = \{\gamma_1, \gamma_2, \ldots, \gamma_k, -\gamma_k, \ldots, -\gamma_2, -\gamma_1\}$ and $k = \frac{p-1}{2d}$ are as above.

Note that when calculating the sum in Proposition 2 we think of the $\gamma_i$ as integers, not integers modulo $p$. For example, we have that $\Gamma_1^6 = \{1, 2, 4, 8, 9, 13, 15, 16\}$ where $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 4$, $\gamma_4 = 8$. Thus,

$$h(X_1^6) \leq \frac{4}{17-1} (1 + 2 + 4 + 8) = 3.75.$$

See Table 1 for some sample calculations of the $\gamma$-bound.

At the very end of the paper we give an upper bound on $h(X_p^m)$ using Proposition 2 and a worse case distribution argument for the sizes of the elements in the first half of $\Gamma_p^m$. The argument leads to the following upper bound:

$$h(X_p^m) \leq \frac{(2d-1)p + (4d - 4d^2 + 1)}{2d^2}. \tag{4}$$

If $m = 2$ and $p \equiv 1 \pmod{4}$, then equation (4) becomes $h(X_p^2) \leq \frac{3p}{8} - \frac{7}{8}$. If $m = 3$ and $p \equiv 1 \pmod{3}$, then equation (4) becomes $h(X_p^3) \leq \frac{5p}{18} - \frac{8}{23}$. Comparing these results to (2) and (3) which were derived from Proposition 1, we see that Proposition 1 gives a better upper bound for the isoperimetric number. However, looking at Table 1 we see that the $\gamma$-bound seems to give a much better upper bound than Proposition 1. Perhaps if one got a better approximation on the worst case for the sizes of the elements in the first half of $\Gamma_p^m$, then one could get a much better explicit upper bound for $h(X_p^m)$ using the $\gamma$-bound. Perhaps this method could lead to an asymptotic formula for $\lim_{p \to \infty} h(X_p^m)/p$ as $p$ goes to infinity for $m > 2$.

2. Basic properties of generalized Paley graphs. In this section we collect together various facts about generalized Paley graphs.

**Proposition 3.** Let $p$ be an odd prime, let $m \geq 2$ be an integer, let $d = \gcd(m, p-1)$. Let $m = 2^na$ for some odd integer $a$ and $n \geq 0$. The following are equivalent.

1. $\Gamma_p^m$ is symmetric.
2. $-1 \in \Gamma_p^m$.
3. $d \mid \frac{p-1}{2}$. 


(4) \( p \equiv 1 (\text{mod} \ 2^{n+1}) \).

**Proof.** (1) \( \iff \) (2): Suppose \( \Gamma^m_p \) is symmetric. Note that \( 1 = 1^m \in \Gamma^m_p \). So, \( -1 \in \Gamma^m_p \). Conversely, suppose \( -1 \in \Gamma^m_p \). Then \( -1 = a^m \) for some \( a \in \mathbb{Z}_p^\times \). If \( x \in \Gamma^m_p \) with \( x = b^m \) and \( b \in \mathbb{Z}_p^\times \), then \( -x = (ab)^m \in \Gamma^m_p \).

(2) \( \iff \) (3): Since \( \mathbb{Z}_p^\times \) is a cyclic group under multiplication there exists \( g \in \mathbb{Z}_p^\times \) where \( \mathbb{Z}_p^\times = \langle g \rangle = \{1, g, g^2, g^3, \ldots, g^{p-2}\} \). Since \( \mathbb{Z}_p \) is a field there are only two solutions to the equation \( x^2 - 1 = (x - 1)(x + 1) = 0 \). These are \( 1 = g^0 \) and \( -1 = g^{(p-1)/2} \). Note that \( -1 \in \Gamma^m_p \) if and only if \( (g^i)^m = g^{(p-1)/2} \) for some integer \( i \) if and only if \( im \equiv \frac{p-1}{2} (\text{mod} \ p-1) \). From [15, pg. 62], \( ax \equiv b (\text{mod} \ m) \) has solutions for \( x \) if and only if \( \gcd(a, m) | b \). Thus, \( im \equiv \frac{p-1}{2} (\text{mod} \ p-1) \) if and only if \( d \mid \frac{p-1}{2} \).

(3) \( \implies \) (4): Suppose that \( d \mid \frac{p-1}{2} \). Recall that \( m = 2^n a \) where \( a \) is odd. Let \( p-1 = 2^k b \) for some \( k, b \in \mathbb{Z} \) where \( b \) is odd. Suppose that \( n \geq k \). Then \( d = \gcd(m, p-1) = \gcd(2^n a, 2^k b) = 2^k c \) for some integer \( c \). But then \( d \) would not divide \( \frac{p-1}{2} = 2^{k-1} b \). Thus \( n < k \). Therefore, \( p \equiv 1 (\text{mod} \ 2^{n+1}) \).

(4) \( \implies \) (2): Suppose that \( m = 2^n a \) where \( a \) is odd. Suppose \( p \equiv 1 (\text{mod} \ 2^{n+1}) \). Then \( p - 1 = 2^{n+1} k \) for some integer \( k \). Since \( \mathbb{Z}_p^\times \) is cyclic there exists \( g \in \mathbb{Z}_p^\times \) where \( \mathbb{Z}_p^\times = \langle g \rangle = \{1, g, g^2, g^3, \ldots, g^{p-2}\} \). If we let \( y = g^{\frac{p-1}{2}} \), then \( y \neq 1 \). Also \( y^2 = g^{p-1} = 1 \). Since \( \mathbb{Z}_p \) is a field the only solutions to \( z^2 - 1 = (z + 1)(z - 1) = 0 \) are \( z = 1 \) and \( z = -1 \). Thus, \( y = g^{\frac{p-1}{2}} = -1 \). If we let \( w = g^{\frac{p-1}{2n}} \), then \( w^m = g^{\frac{p-1}{2n} a} = g^{\frac{p-1}{2} a} = (-1)^a = -1 \). So \( -1 \in \Gamma^m_p \). \( \square \)

The proof of the following lemma is left to the reader.

**Lemma 4.** Let \( m \) be a positive integer, \( p \) be an odd prime, and \( d = \gcd(m, p-1) \). Define the function \( \phi : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \) by \( \phi(x) = x^m \). Then \( \phi \) is \( d \)-to-one.

The facts in the following proposition are proved in [1] for odd \( m \), but in a different way than we present here. [1] also discusses the case where \( p \) is not prime, in which case the generalized Paley graph can sometimes be disconnected.

**Proposition 5.** Let \( m \) be a positive integer, \( p \) be an odd prime, and \( d = \gcd(m, p-1) \) where \( d \mid \frac{p-1}{2} \). Then

(1) \( |\Gamma^m_p| = \frac{p-1}{d} \).
(2) $X^m_p$ is $\frac{p-1}{d}$-regular.

(3) $X^m_p$ is connected.

(4) $X^m_p$ is the complete graph iff $d = 1$.

(5) $X^m_p$ is the cycle graph iff $d = \frac{p-1}{2}$.

Proof. Let $\phi$ be the group homomorphism of $\mathbb{Z}^\times_p$ from Lemma 4. By Lemma 4 we have that $|\Gamma^m_p| = |\mathbb{Z}^\times_p|/|\ker(\phi)| = \frac{p-1}{d}$.

$X^m_p$ is $\frac{p-1}{d}$-regular since $X^m_p$ is a Cayley graph. This implies that $X^m_p$ is complete iff $d = 1$. Similarly this shows that $X^m_p$ is the cycle graph iff $d = \frac{p-1}{2}$.

It is easy to see that $X^m_p$ is connected: since $1 \in \Gamma^m_p$ we have that the cycle with vertices $0, 1, 2, 3, \ldots, p-2, p-1, 0$ is contained in the graph. $\Box$

We now mention a special case of a well-studied problem for Cayley graphs. Two Cayley graphs on the same group $\text{Cay}(G, S)$ and $\text{Cay}(G, T)$ are isomorphic if there exists a group isomorphism $\sigma$ of $G$ such that $S^\sigma = T$. A group $G$ is called a CI-group if the converse is also true. See the survey [13] for more information. It was shown in [5] that $\mathbb{Z}_p$ is a CI-group. The following two propositions are special cases of these results and the proofs are left to the reader.

Proposition 6. Let $p$ be a prime and $m_1 > 1$ and $m_2 > 1$ be integers. Suppose that $d_1 = \gcd(m_1, p-1)$ and $d_2 = \gcd(m_2, p-1)$ and both $d_1$ and $d_2$ divide $\frac{p-1}{2}$. Then $X^{m_1}_p$ and $X^{m_2}_p$ are isomorphic as graphs if and only if $d_1 = d_2$. In particular, note that $X^{d_1}_p$ is isomorphic to $X^{m_1}_p$.

Proposition 7. Let $p$ be a fixed odd prime. A complete list of non-isomorphic generalized Paley graphs of size $p$ is given by the graphs $X^m_p$ where $m$ is a divisor of $(p-1)/2$.

3. An eigenvalue lower bound. In this section we give a proof of the lower bound given in Proposition 1. Throughout this section, let $p$ be an odd prime, $m > 1$ be an integer, and $d = \gcd(p-1, m)$. Assume that $d$ divides $(p-1)/2$.

Since $X^m_p$ is a connected regular graph, by [11, pg. 12] we know that the eigenvalues of $X^m_p$ are real. The next Proposition shows that the eigenvalues of $X^m_p$ are essentially Gauss sums and we give an upper bound for them.
Proposition 8. Let $e_p(x) = e^{2\pi i x/p}$. The eigenvalues of $X^m_p$ are given by

$$\lambda_a = \frac{1}{d} \left( \sum_{n=0}^{p-1} e_p(an^m) - 1 \right), \text{ where } a = 0, 1, \ldots, p - 1.$$

One has $\lambda_0 = \frac{p - 1}{d}$. In addition for each $a = 1, 2, \ldots, p - 1$ we have that

$$\lambda_a \leq \frac{(m - 1)\sqrt{p} - 1}{d}.$$

Proof. Let $\lambda$ be an eigenvalue of $X^m_p$. There exists $0 \leq a \leq p - 1$ where $\lambda = \sum_{\gamma \in \Gamma^m_p} e_p(a\gamma)$. (See [3, pg. 183] or [11, pg. 195].) If $a = 0$, the formula follows. Suppose that $0 < a$. By Lemma 4, for each $\gamma \in \Gamma^m_p$, $|\phi^{-1}(\gamma)| = d$, and so there are $d$ elements $x_1, x_2, \ldots, x_d$ in $\mathbb{Z}_p^\times$ for which $\phi(x_i) = \gamma$. So,

$$\lambda = \sum_{\gamma \in \Gamma^m_p} e_p(a\gamma) = \frac{1}{d} \sum_{n=1}^{p-1} e_p(an^m).$$

Note that

$$\frac{1}{d} \sum_{n=1}^{p-1} e_p(an^m) = \frac{1}{d} \left( \sum_{n=0}^{p-1} e_p(an^m) - 1 \right).$$

The Gauss sum $\sum_{n=0}^{p-1} e_p(an^m)$ can be bounded above (see [9, pg. 1]) by

$$\sum_{n=0}^{p-1} e_p(an^m) \leq (m - 1)\sqrt{p}.$$ 

Therefore,

$$\lambda = \frac{1}{d} \left( \sum_{n=0}^{p-1} e_p(an^m) - 1 \right) \leq \frac{(m - 1)\sqrt{p} - 1}{d}. \quad \Box$$

We now give a proof of the lower bound given in Proposition 1. A known inequality (see [2] and [6], or [11, pg. 31]) tells us that if $X$ is a k-regular graph, then

$$\frac{k - \lambda_1(X)}{2} \leq h(X),$$

(5)
where $\lambda_1(X)$ is the second largest eigenvalue of $X$. Since $d = \gcd(m, p - 1) = \gcd(d, p - 1)$, $X_p^m$ and $X_p^d$ must be isomorphic graphs by Proposition 6, and so by Proposition 8 we have that

$$\lambda_1(X_p^m) = \lambda_1(X_p^d) \leq \frac{(d-1)\sqrt{p}-1}{d}.$$  

We know $X_p^m$ is $\frac{p-1}{d}$-regular by Proposition 5, so combining equations (5) and (6) gives us that

$$h(X_p^m) \geq \frac{p-1}{d} - \frac{(d-1)\sqrt{p}-1}{d} = \frac{p + (1-d)\sqrt{p}}{2d}.  $$

4. An upper bound by estimating solutions to $x^m + y^m + z^m = 0$ modulo $p$. In this section we give a proof of the upper bound given in Proposition 1. Throughout this section, let $p$ be an odd prime, $m > 1$ be an integer, and $d = \gcd(p-1, m)$. Assume that $d$ divides $(p-1)/2$. Let $\Gamma = \Gamma_p^m$. Let $\overline{\Gamma} = \mathbb{Z}_p^\times \setminus \Gamma$.

Note that if $F$ is a subset of $\mathbb{Z}_p$ with $|F| \leq p/2$ then $h(X_p^m) \leq |F|/|\Gamma|$. Therefore, $h(X_p^m) \leq |\overline{\Gamma}|/|\Gamma|$. We now estimate this ratio to get an upper bound on $h(X_p^m)$. We assume that $d > 1$ so that $|\Gamma| < p/2$. (Note that if $d = 1$, then $X_p^m$ is the complete graph whose isoperimetric number is known.)

We first take a look at the structure of $X_p^m$. Please refer to Figure 2 during this discussion. Note that by the definition of a generalized Paley graph, a vertex $x$ is adjacent to 0 if and only if $x \in \Gamma$. Recall that every element of $\Gamma$ has degree $(p-1)/d$. Therefore, we have the following equation

$$|\Gamma| \cdot \frac{p-1}{d} = |\overline{\Gamma}| + 2(\text{number of edges internal to } \Gamma),$$

where the term on the left side of the equation counts one for each endpoint of an edge that lands in $\Gamma$. There is a 2 on the right side of the equation is because each edge that is internal to $\Gamma$ is counted twice on the left side of the equation.

Let

$$S = \{(x, y, z)|x, y, z \in \mathbb{Z}_p^\times \text{ and } x^m + y^m + z^m = 0\}$$

and $N$ denote the size of $S$. Using the facts that every element of $\Gamma$ is an $m$-th power, $-1 \in \Gamma$, and every element of $\Gamma$ can be represented by exactly $d$ different elements of the form $a^m$ where $a \in \mathbb{Z}_p^\times$, one can derive that

$$2(\text{number of edges internal to } \Gamma) = \frac{|S|}{d^3} = \frac{N}{d^3}. $$
Since $|\Gamma| = (p - 1)/d$, we have that equation (7) becomes

$$|\partial \Gamma| = \left(\frac{p - 1}{d}\right)^2 - \frac{N}{d^3}. \tag{8}$$

We now go about estimating the value of $N$.

Let

$$\hat{S} = \{(x, y, z) | x, y, z \in \mathbb{Z}_p \text{ and } x^m + y^m + z^m = 0\}$$

and $\hat{N}$ denote the size of $\hat{S}$. Then $\hat{N} = N + 1 + 3|T|$ where $T = \{(a, b) | a, b \in \mathbb{Z}_p^\times \text{ and } a^m + b^m = 0\}$. Suppose that $a, b \in \mathbb{Z}_p^\times$ with $a^m + b^m = 0$ and $-1 = u^m$ where $u \in \mathbb{Z}_p^\times$. Then $a^m = (ub)^m$. Conversely if $b \in \mathbb{Z}_p^\times$ then by Lemma 4 there are exactly $d$ elements $a \in \mathbb{Z}_p^\times$ with $a^m = (ub)^m$. Thus, $|T| = d(p - 1)$. Therefore,

$$N = \hat{N} - 1 - 3d(p - 1).$$

Weil [18] showed that there exists a positive integer $M_m$, depending only on $m$, with $|\hat{N} - p^2| \leq M_m(p - 1)p^{1/2}$. Moreover, he showed that $M_m$ is the number of triples of rational numbers $(\alpha_0, \alpha_1, \alpha_2)$ satisfying the three conditions

$$m\alpha_0, m\alpha_1, m\alpha_2 \in \mathbb{Z}, \quad \alpha_0 + \alpha_1 + \alpha_2 \in \mathbb{Z}, \quad \text{and} \quad 0 < \alpha_0, \alpha_1, \alpha_2 < 1.$$  

(Another derivation of the above is given as Theorem 5 in [10, pgs. 102–103], however they give a different way to calculate $M_m$. ) This gives that

$$N \geq p^2 - M_m p^{3/2} + M_m p^{1/2} - 1 - 3d(p - 1).$$
Plugging this into equation (8) yields

\[ h(X_p^m) \leq \frac{\lvert \partial \Gamma \rvert}{\lvert \Gamma \rvert} \leq \left( \frac{p - 1}{d} \right)^2 - \frac{p^2 - M_m p^{3/2} + M_m p^{1/2} - 1 - 3d(p - 1)}{d^3}. \]

Simplifying the above equation yields the upper bound given in Proposition 1.

5. The \( \gamma \)-bound. In this section we give a proof of Proposition 2 and a derivation of equation (4). Throughout this section, let \( p \) be an odd prime, \( m > 1 \) be an integer, and \( d = \gcd(p - 1, m) \). Assume that \( d \) divides \( (p - 1)/2 \).

We generalize the \( \alpha \)-bound from [4] (where \( m = 2 \)) to generalized Paley graphs \( (m \geq 2) \). The proof is almost exactly the same, however we have considerably rewritten the details of the proof to make it easier to understand and to simplify it. We have changed the name to the \( \gamma \)-bound as we use \( \gamma \) instead of \( \alpha \) in this version of the proof.

Note that for each \( \gamma \in \mathbb{Z}_p \) we have that \( -\gamma = p - \gamma \). Recall that \( \lvert \Gamma_p^m \rvert = \frac{p - 1}{d} \). Since \( \Gamma_p^m \) is symmetric we may arrange its elements in increasing order.

For the remainder of this section we will use the following notation

\[ \Gamma_p^m = \{ \gamma_1, \gamma_2, \ldots, \gamma_k, -\gamma_k, \ldots, -\gamma_2, -\gamma_1 \} \]

where \( k = \frac{p - 1}{2d} \) and \( 1 \leq \gamma_i \leq \frac{p - 1}{2} \). Note that \( \gamma_1 = 1 \) and \( -\gamma_1 = p - 1 \).

The following is what we call the adjacency table for \( X_p^m \). The top row lists the vertices of \( X_p^m \) and below each vertex \( v \) is a column that contains the vertices that \( v \) is adjacent to.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>p - 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td>\gamma_2</td>
<td>\gamma_2 + 1</td>
<td>\gamma_2 + 2</td>
<td>\ldots</td>
<td>\gamma_2 - 1</td>
</tr>
<tr>
<td>\gamma_3</td>
<td>\gamma_3 + 1</td>
<td>\gamma_3 + 2</td>
<td>\ldots</td>
<td>\gamma_3 - 1</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
</tr>
<tr>
<td>\gamma_k</td>
<td>\gamma_k + 1</td>
<td>\gamma_k + 2</td>
<td>\ldots</td>
<td>\gamma_k - 1</td>
</tr>
<tr>
<td>-\gamma_k</td>
<td>-\gamma_k + 1</td>
<td>-\gamma_k + 2</td>
<td>\ldots</td>
<td>-\gamma_k - 1</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
</tr>
<tr>
<td>-\gamma_3</td>
<td>-\gamma_3 + 1</td>
<td>-\gamma_3 + 2</td>
<td>\ldots</td>
<td>-\gamma_3 - 1</td>
</tr>
<tr>
<td>-\gamma_2</td>
<td>-\gamma_2 + 1</td>
<td>-\gamma_2 + 2</td>
<td>\ldots</td>
<td>-\gamma_2 - 1</td>
</tr>
<tr>
<td>p - 1</td>
<td>0</td>
<td>1</td>
<td>\ldots</td>
<td>p - 2</td>
</tr>
</tbody>
</table>
We will now use the adjacency table for $X^m_p$ to get a bound on $h(X^m_p)$. We will do this by considering a special set. This set is $F = \{0, 1, 2, \ldots , \frac{p-3}{2}\}$. We will use the table to get a formula for the size of $\partial F$. Once we have done this we will be able to bound $h(X^m_p)$ by using the formula $h(X^m_p) \leq |\partial F|/|F|$. 

**Proposition 9.** Let $F = \{0, 1, 2, \ldots , \frac{p-3}{2}\}$ be as above. Consider the row of the adjacency table for $X^m_p$ that begins with $\gamma_i$ where $1 \leq i \leq k$. The row corresponding to $\gamma_i$ of the adjacency table for $X^m_p$ contributes exactly $\gamma_i$ edges to the boundary set $\partial F$.

**Proof.** When looking at the row corresponding to $\gamma_i$ we need to scan the entries from the column with header 0 to the column with header $\frac{p-3}{2}$ and count how many entries are greater than $\frac{p-3}{2}$. Each entry that we find that is greater than $\frac{p-3}{2}$ will contribute an edge to $\partial F$. Notice that as we move from left to right in the table each entry increases by one each time we move right.

Suppose we are in the row corresponding to $\gamma_i$. We break the proof into two cases: $\gamma_i \neq \frac{p-1}{2}$ and $\gamma_i = \frac{p-1}{2}$.

We begin with the first case. Assume that $\gamma_i \neq \frac{p-1}{2}$. Then $\gamma_i \leq \frac{p-3}{2}$. Suppose we start at the column with header 0 and scan one by one to the right until we arrive at the entry $\frac{p-3}{2}$ in some column $\beta$. So far we have not found any entries that contribute an edge to $\partial F$. Since $\frac{p-3}{2}$ is in row $\gamma_i$ and column $\beta$ we have that $\frac{p-3}{2} = \gamma_i + \beta$. So, $\beta = \frac{p-3}{2} - \gamma_i$. Scanning from column 0 to column $\beta$ we have encountered $\beta + 1$ entries. Note that $|F| = \frac{p-1}{2}$. So we have exactly

$$\frac{p-1}{2} - (\beta + 1) = \frac{p-1}{2} - \left(\frac{p-3}{2} - \gamma_i + 1\right) = \gamma_i$$

entries left to consider in our scan of columns headed by elements of $F$. The remaining entries in the table consist of the elements $\frac{p-3}{2} + 1$ to $\frac{p-3}{2} + \gamma_i$. By definition we know that $\gamma_i \leq \frac{p-1}{2}$. Therefore $\frac{p-3}{2} + \gamma_i \leq p-2 < p$. Hence as we scan the remaining entries we never pass $p-1$ and cycle back to 0. Therefore, each of the remaining $\gamma_i$ entries contributes an edge to $\partial F$, which is what we
wanted to prove.

Suppose that we are in the second case, that is \( \gamma_i = \frac{p-1}{2} \). Then every element in row \( \gamma_i \) starting from column 0 to column \( \frac{p-3}{2} \) corresponds to an element that is not in \( F \) and hence contributes an edge to \( \partial F \). This gives us \( \gamma_i = \frac{p-1}{2} \) entries that contribute and edge to \( \partial F \). \( \square \)

We now prove that the adjacency table has a symmetric property.

**Proposition 10.** \( F = \left\{ 0, 1, 2, \ldots, \frac{p-3}{2} \right\} \) be as above. The row beginning with entry \( -\gamma_i \) of the adjacency table for \( X_p^m \) contributes the same number of edges to \( \partial F \) as does the row beginning with entry \( \gamma_i \).

**Proof.** Consider the row corresponding to \( -\gamma_i \). As in Proposition 9, we only need to inspect the entries from column 0 to column \( \frac{p-3}{2} \). As we do this we count the number of entries that are greater than \( \frac{p-3}{2} \).

Suppose first that \( \gamma_i = \frac{p-1}{2} \). Then in this case, \( -\gamma_i = \frac{p+1}{2} \). Thus, every element from column 0 to column \( \frac{p-3}{2} \) corresponds to an element that is not in \( F \). Hence, we count \( |F| = \frac{p+1}{2} = \gamma_i \) entries that correspond to edges in \( \partial F \).

Now suppose that \( 1 \leq \gamma_i \leq \frac{p-3}{2} \) and again consider the row corresponding to \( -\gamma_i \). We start in the column with 0 and scan until we reach \( p-1 \) in some column headed with say \( \beta \). Doing this we have counted \( \beta+1 \) entries. Since \( p-1 \) is in row \( -\gamma_i \) and column \( \beta \) we have that \( p-1 = -\gamma_i + \beta \). This gives us that \( \beta + 1 = \gamma_i \). So we have scanned exactly \( \gamma_i \) entries so far. Since \( \beta = \gamma_i - 1 \) and \( 1 \leq \gamma_i \leq \frac{p-3}{2} \) we know that \( 0 \leq \beta \leq \frac{p-5}{2} \). That is, \( \beta \) is an element of \( F \) and we still have at least one more entry in our row to scan to the right of column \( \beta \). Since we have scanned exactly \( \gamma_i \) entries so far, the remaining entries to scan correspond to entry 0 in column \( \beta+1 \) to entry \((p-1) + \left( \frac{p-1}{2} - \gamma_i \right) = \frac{p-3}{2} - \gamma_i \leq \frac{p-5}{2} \) in column \( \frac{p-3}{2} \). Thus the remaining entries are all in \( F \) and hence do not contribute any edges to \( \partial F \). Therefore, in this case we get exactly \( \gamma_i \) entries in row \( -\gamma_i \) that correspond to an edge in \( \partial F \). \( \square \)
Proposition 11. Let \( F = \left\{ 0, 1, 2, \ldots, \frac{p - 3}{2} \right\} \) and
\[
\Gamma^m_p = \{ \gamma_1, \gamma_2, \ldots, \gamma_k, -\gamma_k, \ldots, -\gamma_2, -\gamma_1 \}
\]
where \( k = \frac{p - 1}{2d} \) as above. Then
\[
|\partial F| = 2 \sum_{i=1}^{k} \gamma_i,
\]

Proof. This follows from Proposition 9 and Proposition 10.

We can now derive the \( \gamma \)-bound given in Proposition 2. Let \( F = \left\{ 0, 1, 2, \ldots, \frac{p - 3}{2} \right\} \). By Proposition 11 and the fact that \( |F| = \frac{p - 1}{2} \) we see that
\[
h(X^m_p) \leq \frac{|\partial F|}{|F|} = \frac{2 \sum_{i=1}^{k} \gamma_i}{\frac{p - 1}{2}} = \frac{4}{p - 1} \sum_{i=1}^{k} \gamma_i.
\]

We now derive equation (4) from the introduction. Recall that \( 1 \leq \gamma_k \leq \frac{p - 1}{2} \), that is, the \( \gamma_k \) are all trapped in the first half of \( \mathbb{Z}_p^\times \). Therefore, the worst that the \( \gamma \)-bound can be is if all the \( \gamma_k \) are grouped together as close as possible to \( (p - 1)/2 \). Since \( \gamma_1 = 1 \), if all the remaining \( \gamma_k \) are grouped up next to \( \frac{p - 1}{2} \) then we would have that \( \sum_{i=1}^{k} \gamma_k \) is less than or equal to
\[
1 + \left( \frac{p - 1}{2} - (k - 2) \right) + \left( \frac{p - 1}{2} - (k - 3) \right) + \cdots + \left( \frac{p - 1}{2} - 1 \right) + \frac{p - 1}{2}.
\]
Using the formula
\[
a + (a + 1) + (a + 2) + \cdots + (a + n) = a(n + 1) + \frac{n(n + 1)}{2}
\]
with \( a = \frac{p - 1}{2} - (k - 2) \) and \( n = k - 2 \) we get that
\[
h(X^m_p) \leq \frac{4}{p - 1} \left( 1 + \left[ \frac{p - 1}{2} - (k - 2) \right] [k - 1] + \frac{(k - 2)(k - 1)}{2} \right)
\]
The isoperimetric number of a generalized Paley graph

\[
= \frac{2d(2 + p) - p - 4d^2 + 1}{2d^2}
= \frac{(2d - 1)p + (4d - 4d^2 + 1)}{2d^2}.
\]

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REFERENCES


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