A trace formula for finite upper half planes

Anthony Shaheen

Department of Mathematics, California State University, Los Angeles,
5151 State University Drive, Los Angeles, California 90032
e-mail: ashahee@calstatela.edu

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Abstract. In this paper, we prove a trace formula for finite upper half planes $H_q$. A brief outline is as follows: Fix a subgroup $\Gamma \subset GL(2, \mathbb{F}_q)$. The adjacency operators $A_\alpha$ act on functions in $L^2(\Gamma \backslash H_q)$; thus, we may consider $A_\Gamma^\alpha = A_\alpha|_{L^2(\Gamma \backslash H_q)}$. We prove a trace formula which is an equality between a weighted sum of the traces of the operators $A_\Gamma^\alpha$ and a sum over the conjugacy classes of $\Gamma$. The trace formula allows us to compute the trace of $A_\Gamma^\alpha$. We compute the trace formula for the subgroups $\Gamma = N$ and $\Gamma = K$ of $GL(2, \mathbb{F}_q)$.

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1. Introduction

Before outlining the results of this paper, we first review finite upper half planes.

The Poincaré upper half plane

$$H = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$$

has a finite analogue: Let $p$ be an odd prime, $q = p^r$ with $r \geq 1$, $\mathbb{F}_q$ be the finite field of $q$ elements, and $\delta$ be a non-square element of $\mathbb{F}_q$. The finite “upper” half plane is defined to be

$$H_q = \{x + y\sqrt{\delta} \mid x, y \in \mathbb{F}_q, y \neq 0\}.$$

A good reference about finite upper half planes is [18]. The smallest finite upper half plane, $H_3$, is shown in Figure 1, where $\delta = -1$, and $y = \sqrt{\delta} = \sqrt{-1}$.  

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Figure 1. $H_3$.

Let $z = x + y\sqrt{\delta}$ be an element of $H_q$. We write $x = Re(z)$ and $y = Im(z)$ for the real and imaginary parts of $z$, $\overline{z} = x - y\sqrt{\delta}$ for the conjugate of $z$, $N(z) = z\overline{z}$ for the norm of $z$, and $Tr(z) = z + \overline{z}$ for the trace of $z$.

An element of the **general linear group**

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{F}_q)$$

has entries in $\mathbb{F}_q$ and non-zero determinant. Given $z \in H_q$, we define

$$gz = \frac{az + b}{cz + d}.$$  

We have that

$$Im(gz) = \frac{Im(z) \det g}{N(cz + d)}. \quad (1.1)$$

Therefore, $gz \in H_q$.

The group $\text{GL}(2, \mathbb{F}_q)$ acts as the finite analogue of $SL(2, \mathbb{R})$. We will frequently let $G$ denote $\text{GL}(2, \mathbb{F}_q)$.

Let

$$K = \left\{ g \in \text{GL}(2, \mathbb{F}_q) \mid g\sqrt{\delta} = \sqrt{\delta} \right\} \quad (1.2)$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}_q, \ a^2 - \delta b^2 \neq 0 \right\}, \text{ and}$$

$$N = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\}. \quad (1.3)$$

There is a correspondence between $H_q$ and $G/K$:

$$G/K \rightarrow H_q$$

$$gK \mapsto g\sqrt{\delta}$$
A set of coset representatives for $G/K$ is given by
\[
\text{Aff}(q) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{F}_q, y \neq 0 \right\}.
\] (1.4)

Define a “distance” between $z, w \in H_q$ by
\[
d(z, w) = \frac{N(z - w)}{\text{Im}(z)\text{Im}(w)}.
\]

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{F}_q)$ and $z, w \in H_q$. Combine
\[
N(gz - gw) = \frac{N(\det(g)(z - w))}{N(cz + d)N(cw + d)} = \frac{\det(g)^2 N(z - w)}{N(cz + d)N(cw + d)}
\]
and (1.1) to get that $d$ is a point-pair invariant; that is,
\[
d(gz, gw) = d(z, w).
\] (1.5)

For $a \in \mathbb{F}_q$, define the finite upper half plane graph $X_q(\delta, a)$ as follows: Let the vertices of the graph $X_q(\delta, a)$ be the points in $H_q$. Connect two vertices $z, w \in H_q$ if and only if $d(z, w) = a$. For $a \neq 0, 4\delta$, the graph $X_q(\delta, a)$ is $(q + 1)$-regular, connected, and Ramanujan.

Ramanujan graphs are good expander graphs and their zeta functions satisfy an analogue of the Riemann Hypothesis. See [11] and [18, pgs. 55–58 and 418].

Consider $f \in L^2(H_q) = \{ g : H_q \to \mathbb{C} \}$. For each $a \in \mathbb{F}_q$, define the adjacency operator for the graph $X_q(\delta, a)$ by the formula
\[
A_a f(z) = \sum_{\substack{w \in H_q \\
d(z, w) = a}} f(w).
\] (1.6)

The adjacency operators form a commutative, $\text{GL}(2, \mathbb{F}_q)$-invariant set of operators.

Let
\[
S_q(\delta, a) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{F}_q, y \neq 0, x^2 = ay + \delta(y - 1)^2 \right\}.
\] (1.7)

Note that $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in S_q(\delta, a)$ if and only if $d(x + y\sqrt{\delta}, \sqrt{\delta}) = a$. Thus, by (1.5), we may rewrite (1.6) as
\[
A_a f(z) = \sum_{s_a \in S_q(\delta, a)} f(gs_a \sqrt{\delta}),
\] (1.8)
Table 1. Cayley graphs and adjacency operators for $H_3$ in terms of the standard basis for $L^2(H_3)$.

<table>
<thead>
<tr>
<th>$X_q(1, 0)$</th>
<th>$X_q(-1, 1)$</th>
<th>$X_q(-1, -1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Cayley Graph 1" /></td>
<td><img src="image2" alt="Cayley Graph 2" /></td>
<td><img src="image3" alt="Cayley Graph 3" /></td>
</tr>
</tbody>
</table>

where $z = x + y\sqrt{\delta}$ and $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$. One can show that the set of $K$-double cosets for $G$ are represented by the sets $S_q(\delta, a)$ for $a \in \mathbb{F}_q$. See [18, pg. 317] for details.

Let us give an example: Let $q = 3, \delta = -1, y = \sqrt{-1}$. Let $\{1+y, y, 1+y, -1+y, -y, 1-y\}$ be the ordering on $H_3$. Table 1 lists the Cayley graphs $X_q(\delta, a)$ and the adjacency operators $A_a$, for $a \in \mathbb{F}_3$, given in terms of the standard basis for $L^2(H_3)$.

In the future we will make the following notational conventions:

$$f(x + y\sqrt{\delta}) = f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\sqrt{\delta}\right) = f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right).$$

The combinatorial Laplacian of a $k$-regular graph $X$ is $\Delta = A - kI$, where $A$ is the adjacency operator of $X$ (see [18, pg. 51]). Note that when studying eigenfunctions and eigenvalues, one may study either $A$ or $\Delta$. One may think of the operators $A_a - k_aI$, where $k_a$ is the degree of $X_a(\delta, a)$, as finite analogs of the non-Euclidean Laplacian on the Poincaré upper half plane $H$. Let me suggest several reasons for this.

A simpler example is worked out in full detail in [18, Ch. 7]. It is shown that the combinatorial Laplacian for the Cayley graph $X(\mathbb{Z}/n\mathbb{Z}, \pm 1)$ approximates the Laplacian of the real line $\frac{d^2}{dx^2}$. In fact, the whole book is devoted to finding finite analogues of harmonic analysis on symmetric spaces found in [16]. There are finite analogs of $k$-Bessel functions and spherical functions, which classically are eigenfunctions of the non-Euclidean Laplacian.
\[ \Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \] on \( \mathbb{H} \), while in the finite case they are eigenfunctions of the adjacency matrices \( A_a \). There are also finite versions of Maass waveforms (see the end of Section 2 in this paper).

The combinatorial Laplacian has been studied extensively by graph theorists. (The definition of \( \Delta \) varies depending on the graph theorist. For example, in [5], the combinatorial Laplacian of a \( k \)-regular graph with adjacency operator \( A \) is defined to be \( \Delta = I - \frac{1}{k} A \). Also, there is a more general definition of \( \Delta \) for non-regular graphs.) There are analogies between the spectral theory of the Laplacian on Riemannian manifolds and graphs. See [5] and the many references given in [18, pg. 116]. [6] explores analogies between the combinatorial Laplacian of a connected infinite graph and the Laplace-Beltrami operator on a complete Riemannian manifold.

The outline of this paper is as follows: We show that the \( A_a \) are operators on the space \( L^2(\Gamma \backslash H_q) = \{ f : H_q \to \mathbb{C} \mid f(\gamma z) = f(z), \forall \gamma \in \Gamma, z \in H_q \} \), where \( \Gamma \) is any subgroup of \( \text{GL}(2, \mathbb{F}_q) \). Next we show that the set of operators \( A^\Gamma_a = A_a \mid_{L^2(\Gamma \backslash H_q)} \) can be simultaneously diagonalized by an orthonormal basis of \( L^2(\Gamma \backslash H_q) \).

In [7], the Selberg trace formula is given for a compact Riemann surface of genus \( g \geq 2 \). In this paper, we follow the ideas of [7] and prove a trace formula for finite upper half planes. Essentially, without introducing all of the details, the trace formula says that

\[
\sum_{a \in \mathbb{F}_q} \Phi(a) Tr(A^\Gamma_a) = \sum_{\{\gamma\}} I(\Phi, \gamma),
\]

where the sum over \( \{\gamma\} \) means the sum over the conjugacy classes of \( \Gamma \), \( I(\Phi, \gamma) \) is a certain sum (an analogue of the integrals from the classical trace formula), and \( \Phi \) is any function from \( \mathbb{F}_q \) to \( \mathbb{C} \). In Remark 3.8, we note the similarities and differences between the trace formula for \( H_q \) and the one given in [7].

We conclude the paper by computing the trace formula for \( \Gamma = N \) and \( \Gamma = K \). The \( \Gamma = N \) computation is straightforward, while the \( \Gamma = K \) computation involves counting points on a quadratic curve over \( \mathbb{F}_q \). We then compute \( Tr(A^\Gamma_n) \) for \( \Gamma = N \) and \( \Gamma = K \) by plugging \( \Phi(a) = \delta_c(a) \) into the trace formula, where

\[
\delta_c(a) = \begin{cases} 
1, & a = c \\
0, & a \neq c
\end{cases}
\]

In [18, Ch. 23], Terras gives a finite trace formula for \( \Gamma \subset \text{GL}(2, \mathbb{F}_q) \), but it is in the language of representation theory.
In Remark 3.9, we note that when $\Gamma = K$, we are finding the traces of elements of the Hecke algebra $H(K, G)$.

2. $A_a^\Gamma$ and the spaces of functions $L^2(\Gamma \backslash H_q)$

In this section, we show that $A_a$ maps $L^2(\Gamma \backslash H_q)$ to $L^2(\Gamma \backslash H_q)$; consequently, we define $A_a^\Gamma = A_a|_{L^2(\Gamma \backslash H_q)}$. We then show that the operators $A_a^\Gamma$ can be simultaneously diagonalized by an orthonormal basis of $L^2(\Gamma \backslash H_q)$. Recall that $G = \text{GL}(2, \mathbb{F}_q)$.

2.1 Some facts about $A_a$ and $L^2(H_q)$

We make $L^2(H_q)$ into an inner product space by $\langle f, g \rangle = \sum_{x \in H_q} f(x)g(x)$.

We treat $L^2(\Gamma \backslash H_q)$ as a subspace of $L^2(H_q)$ and use the same inner product.

Lemma 2.1. $A_a$ and $A_b$ commute as operators on $L^2(H_q)$.

Proof. This is given as a Corollary in [2, pg. 17].

Lemma 2.2. $A_a$ is an operator on $L^2(\Gamma \backslash H_q)$.

Proof. Let $f \in L^2(\Gamma \backslash H_q)$, $\gamma \in \Gamma$, $z = x + y\sqrt{\delta} \in H_q$, and $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$. Then,

$$ (A_a f)(\gamma z) = (A_a f)(\gamma g\sqrt{\delta}) $$

$$ = \sum_{s \in S_q(\delta, a)} f(\gamma gs\sqrt{\delta}) = \sum_{s \in S_q(\delta, a)} f(gs\sqrt{\delta}) = (A_a f)(z). $$

Therefore, $Af \in L^2(\Gamma \backslash H_q)$. [qed]

Definition 2.3. By Lemma 2.2, we can define

$$ A_a^\Gamma = A_a|_{L^2(\Gamma \backslash H_q)} : L^2(\Gamma \backslash H_q) \to L^2(\Gamma \backslash H_q). $$

Remark 2.4. It can be shown that $A_a$ is self adjoint with respect to the inner product given on $L^2(H_q)$. See [18, pg. 52]. Since the $A_a$ are self-adjoint, commuting operators on $L^2(\Gamma \backslash H_q)$, so are the $A_a^\Gamma$.

Remark 2.5. The spectrum of $A_a^\Gamma$ is contained in the spectrum of $A_a$.

2.2 Eigenfunctions of $A_a^\Gamma$

We make use of the following theorem from [8, pg. 52]: Let $F \subset M_n(\mathbb{C})$ be a family of diagonalizable matrices. Then $F$ is a commuting family if and only if it is a simultaneously diagonalizable family.
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−1+y  y  1+y

−1−y  −y  1−y

Figure 2. A fundamental domain for \( N\backslash H_3 \).

Proposition 2.6. There exists an orthonormal basis of eigenfunctions of \( L^2(\Gamma\backslash H_q) \) that simultaneously diagonalize all the \( A^\Gamma_a \).

Proof. Let \( F = \{ A_a \mid a \in \mathbb{F}_q \} \). Use Remark 2.4, the spectral theorem for self-adjoint operators, and the theorem stated above from [8]. \( \square \)

Definition 2.7. We define the automorphic functions on \( \Gamma\backslash H_q \) to be

\[
\text{Aut}(\Gamma\backslash H_q) = \{ f \in L^2(\Gamma\backslash H_q) \mid f \text{ is an eigenfunction of all the } A^\Gamma_a \}.
\]

Proposition 2.6 tells us that we can find an orthonormal basis for \( L^2(\Gamma\backslash H_q) \) whose elements are eigenfunctions of all the \( A^\Gamma_a \); that is, we can find an orthonormal basis for \( L^2(\Gamma\backslash H_q) \) sitting in \( \text{Aut}(\Gamma\backslash H_q) \).

Example 2.8. Let \( q = 3, \delta = -1, \) and \( y = \sqrt{-1} = \sqrt{-1} \). For all our matrices and vectors in this example we will use the following ordering on \( H_3 \): \( \{-1 + y, y, 1 + y, -1 - y, -y, 1 - y\} \). Let \( \Gamma = N = \{(1,0,0,1), (1,1,1,0), (1,2,0,1)\} \) (see (1.3) for a definition of \( N \)). Then, \( \{y, -y\} \) is a fundamental domain for \( N\backslash H_3 \). A picture of the fundamental domain for \( N\backslash H_3 \) is given in Figure 2.

The matrices for the adjacency operators in terms of the standard basis for \( L^2(H_3) \) are given below (using the ordering of \( H_3 \) given above):

\[
A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}
\]
The eigenvalues of $A_0, A_1$, and $A_{-1}$ are $\lambda = 1(6)$, $\lambda = 4, -2(2), 0(3)$, and $\lambda = -1(3), 1(3)$, respectively, where if we write $1(6)$ we mean $1$ with multiplicity $6$.

The basis $\left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$ of $L^2(N \backslash H_q)$ gives an orthonormal set of eigenfunctions for the adjacency operators. Their matrices in terms of this basis are $A^N_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^N_1 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$, and $A^N_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

One can study elements of the space $L^2(\Gamma \backslash H_q)$ by considering Eisenstein series. When $\Gamma = K$, the Eisenstein series are called spherical functions. See [18, Ch. 20] for more information about spherical functions on finite upper half planes. The Eisenstein series for $\Gamma = \text{SL}(2, \mathbb{F}_p)$ and $\Gamma = \text{GL}(2, \mathbb{F}_p)$ are studied in [15]. They have Fourier expansions that are similar to those of Maass waveforms for $\text{SL}(2, \mathbb{Z})$ that can be found in [16, pg. 208]. One can also imitate holomorphic modular forms. See [14].

3. A trace formula for $H_q$

In this section, we follow [7] and give a trace formula for $H_q$. This trace formula differs from the trace formula in [18]: it involves a weighted sum over the traces of the operators $A^\Gamma_a$, while the trace formula in [18] involves a sum over the representations occurring in $\text{Ind}_G^G(1)$.

As before, $G$ denotes the group $\text{GL}(2, \mathbb{F}_q)$, and $\Gamma$ is a subgroup of $G$.

3.1 The operator $L_\Phi$

We start by defining an operator on $L^2(\Gamma \backslash H_q)$ that shares the same eigenfunctions as the $A^\Gamma_a$.

**Definition 3.1.** Let $\Phi : \mathbb{F}_q \to \mathbb{C}$ be any function and let $f \in L^2(\Gamma \backslash H_q)$. Define the operator $L_\Phi : L^2(\Gamma \backslash H_q) \to L^2(\Gamma \backslash H_q)$ by

$$(L_\Phi f)(z) = \sum_{w \in H_q} \Phi(d(z, w)) f(w).$$
Remark 3.2. Let $f \in L^2(\Gamma \backslash H_q)$. Then,
\[
(L_\Phi f)(z) = \sum_{w \in H_q} \Phi(d(z, w)) f(w)
\]
\[
= \sum_{a \in \mathbb{F}_q} \sum_{w \in H_q, d(z, w) = a} \Phi(a) f(w)
\]
\[
= \sum_{a \in \mathbb{F}_q} \Phi(a) (A^\Gamma_a f)(z).
\]

Hence, $L_\Phi$ is a weighted sum of the adjacency operators $A^\Gamma_a$.

Lemma 2.2 and Remark 3.2 tell us that $L_\Phi$ is a well-defined map from $L^2(\Gamma \backslash H_q)$ to $L^2(\Gamma \backslash H_q)$.

Proposition 3.3. If $f$ is an eigenfunction of all the $A^\Gamma_a$, then $f$ is an eigenfunction of $L_\Phi$. More precisely, if $f \in \text{Aut}(\Gamma \backslash H_q)$ with $A^\Gamma_a f = \lambda_a f$ for all $a \in \mathbb{F}_q$, then $f$ is an eigenfunction of $L_\Phi$ and
\[
(L_\Phi f)(z) = \left( \sum_{a \in \mathbb{F}_q} \Phi(a) \lambda_a \right) f(z).
\]

Proof. By Remark 3.2,
\[
(L_\Phi f)(z) = \sum_{a \in \mathbb{F}_q} \Phi(a) (A^\Gamma_a f)(z) = \left( \sum_{a \in \mathbb{F}_q} \Phi(a) \lambda_a \right) f(z).
\]

As in [7], we make the following definition.

Definition 3.4. Define
\[
k(z, w) = \Phi(d(z, w))
\]
and
\[
K(z, w) = \sum_{\gamma \in \Gamma} \Phi(d(\gamma z, w)) = \sum_{\gamma \in \Gamma} k(\gamma z, w).
\]

Lemma 3.5.
(1) $k(gz, w) = k(z, g^{-1} w)$ for all $g \in G$.
(2) $K(z, w) = K(w, z)$ for all $z, w \in H_q$. 

(3) \( K(z, w) = \sum_{\gamma \in \Gamma} k(\gamma z, w) = \sum_{\gamma \in \Gamma} k(z, \gamma w). \)

(4) \( K(\gamma_1 z, \gamma_2 w) = K(z, w) \) for all \( z, w \in H_q \) and \( \gamma_1, \gamma_2 \in \Gamma. \)

Proof. Part 1 follows from \( d(gz, w) = d(z, g^{-1} w). \) Using part 1, we get part 2 and 3 from

\[
K(z, w) = \sum_{\gamma \in \Gamma} k(\gamma z, w) = \sum_{\gamma \in \Gamma} k(z, \gamma^{-1} w) = \sum_{\gamma \in \Gamma} k(z, \gamma w) = K(w, z).
\]

Part 4 is proven by

\[
K(\gamma_1 z, \gamma_2 w) = \sum_{\gamma \in \Gamma} k(\gamma_1 z, \gamma_2 w) = \sum_{\gamma \in \Gamma} k(\gamma^{-1}_2 \gamma_1 z, w) = \sum_{\gamma \in \Gamma} k(\gamma z, w) = K(z, w).
\]

\[\square\]

We want to analyze \( L_\Phi \) using the geometry of \( \Gamma \backslash H_q. \) Suppose that

\[
\Gamma \backslash H_q = \{ \Gamma x_1, \Gamma x_2, \ldots, \Gamma x_n \}, \tag{3.1}
\]

where \( n = |\Gamma \backslash H_q|, x_i \in H_q, \) and \( H_q = \coprod_{i=1}^n \Gamma x_i. \) (We will see later, in some examples, that the orbits \( \Gamma x_i \) may differ in size.) From our definitions we have

\[
(L_\Phi f)(z) = \sum_{w \in H_q} k(z, w) f(w).
\]

Let

\[
Stab_\Gamma(x) = \{ \gamma \in \Gamma \mid \gamma x = x \}
\]

be the stabilizer of \( x \) in \( \Gamma. \) In the following proof, we make use of \( Stab_\Gamma(x) \) since we have duplication in our sum if we have elliptic or central terms of \( GL(2, \mathbb{F}_q) \) in \( \Gamma \) (see table 2).

**Lemma 3.6.** If \( f \in L^2(\Gamma \backslash H_q) \) and \( \Gamma \backslash H_q \) is decomposed as in (3.1), then

\[
(L_\Phi f)(z) = \sum_{\Gamma x_i \in \Gamma \backslash H_q} \left( \frac{1}{|Stab_\Gamma(x_i)|} K(z, x_i) \right) f(x_i).
\]
Proof. Since $|\text{Stab}_\Gamma(x)| = |\text{Stab}_\Gamma(\gamma x)|$ for any $\gamma \in \Gamma$,

$$(L_\Phi f)(z) = \sum_{w \in \mathcal{H}} k(z, w) f(w) = \sum_{i=1}^{n} \sum_{\gamma \in \Gamma} \frac{1}{|\text{Stab}_\Gamma(x_i)|} k(z, \gamma x_i) f(\gamma x_i)$$

$$= \sum_{i=1}^{n} \sum_{\gamma \in \Gamma \cap \mathcal{H}} \left( \frac{1}{|\text{Stab}_\Gamma(x_i)|} K(z, x_i) \right) f(x_i).$$

We are using Lemma 3.5, part (3). \hfill \Box

Using Lemma 3.6, we see that the matrix for $L_\Phi : L^2(\Gamma \setminus \mathcal{H}_q) \to L^2(\Gamma \setminus \mathcal{H}_q)$ in terms of the standard basis for $L^2(\Gamma \setminus \mathcal{H}_q)$ is

$$
\begin{pmatrix}
\frac{1}{|\text{Stab}_\Gamma(x_1)|} K(x_1, x_1) & \frac{1}{|\text{Stab}_\Gamma(x_2)|} K(x_1, x_2) & \cdots & \frac{1}{|\text{Stab}_\Gamma(x_n)|} K(x_1, x_n) \\
\frac{1}{|\text{Stab}_\Gamma(x_1)|} K(x_2, x_1) & \frac{1}{|\text{Stab}_\Gamma(x_2)|} K(x_2, x_2) & \cdots & \frac{1}{|\text{Stab}_\Gamma(x_n)|} K(x_2, x_n) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{|\text{Stab}_\Gamma(x_1)|} K(x_n, x_1) & \frac{1}{|\text{Stab}_\Gamma(x_2)|} K(x_n, x_2) & \cdots & \frac{1}{|\text{Stab}_\Gamma(x_n)|} K(x_n, x_n)
\end{pmatrix}.
$$

(3.2)

3.2 The trace formula

We now take the trace of $L_\Phi$.

**Theorem 3.7. (Trace Formula).** If $\Gamma$ is a subgroup of $\text{GL}(2, \mathbb{F}_q)$, $\Phi : \mathbb{F}_q \to \mathbb{C}$ is any function, and $\phi_1, \ldots, \phi_n$ are an orthonormal basis of $L^2(\Gamma \setminus \mathcal{H}_q)$ consisting of eigenfunctions of the $A_\Gamma^a$ for all $a \in \mathbb{F}_q$ (from Proposition 2.6), then

1. $\phi_1, \ldots, \phi_n$ are eigenfunctions of $L_\Phi$,
2. $$(L_\Phi \phi_i)(z) = \left( \sum_{a \in \mathbb{F}_q} \Phi(a) \lambda_a^{(i)} \right) \phi_i(z),$$

where $(A_\Gamma^a \phi_i)(z) = \lambda_a^{(i)} \phi_i(z)$, and
3. $\text{Tr}(L_\Phi) = \sum_{a \in \mathbb{F}_q} \Phi(a) \text{Tr}(A_\Gamma^a)$

$$= \sum_{i=1}^{n} \left( \sum_{a \in \mathbb{F}_q} \Phi(a) \lambda_a^{(i)} \right) = \sum_{\{\gamma\}} I(\Phi, \gamma), \quad (3.3)$$
where the sum over \( \{ \gamma \} \) means the sum over the conjugacy classes of \( \Gamma \), \( I(\Phi, \gamma) = \sum_{\gamma y_i \in \Gamma \gamma \setminus H} \frac{1}{|Stab_{\gamma}(y_i)|} \Phi^i(d\gamma y_i, y_i) \), and \( \Gamma \gamma = \{ \gamma' \in \Gamma \mid \gamma' \gamma = \gamma' \} \).

**Proof.** Part 1 and 2 follow from Proposition 3.3. Taking the trace of the matrix in (3.2), we have

\[
Tr(L_\Phi) = \sum_{i=1}^{n} \frac{1}{|Stab_{\gamma}(x_i)|} K(x_i, x_i) = \sum_{\gamma x_i \in \Gamma \setminus H} \frac{1}{|Stab_{\gamma}(x_i)|} \sum_{\gamma \in \Gamma} k(\gamma x_i, x_i) = \sum_{\gamma x_i \in \Gamma \setminus H} \frac{1}{|Stab_{\gamma}(x_i)|} \sum_{\gamma \in \Gamma} \sum_{\gamma' \in \Gamma} k(\gamma x_i, \gamma x_i) = \sum_{\gamma' \gamma x_i \in \Gamma \setminus H} \frac{1}{|Stab_{\gamma}(y_i)|} k(\gamma y_i, y_i).
\]

We used the fact that \( |Stab_{\gamma}(x_i)| = |Stab_{\gamma}(x_i)| \).

**Remark 3.8.** We compare this trace formula to the one given in [7, Ch. 1]. Let \( F \) denote a compact Riemann surface of genus \( g \geq 2 \). One may represent \( F \) as a quotient space \( \Gamma \setminus H \), where \( \Gamma \) is a strictly hyperbolic Fuchsian group and \( H \) is the Poincaré upper half plane. The Laplace-Beltrami operator for \( F \) corresponds to the Laplacian \( \Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) for \( H \). One can find an orthonormal basis \( \{ \phi_\alpha \}_{\alpha=0}^\infty \) for \( L_2(\Gamma \setminus H) \) corresponding to eigenvalues \( 0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \cdots \) of \( \Delta \), such that \( \lim_{n \to \infty} \lambda_n = -\infty \). A preliminary version of Selberg’s trace formula [7, pg. 24] is given by:

\[
\sum_{i=1}^{\infty} \Lambda(\lambda_i) = \sum_{\{ T \}} \int_{F(R)} k(Tz, z) d\mu(z) \tag{3.4}
\]

where the sum over \( \{ T \} \) means the sum over distinct conjugacy classes in \( \Gamma \), \( Z(T) \) is the centralizer of \( T \) in \( \Gamma \), \( FR(Z(T)) \) is a “reasonable” fundamental region in \( H \) for the Fuchsian subgroup \( Z(T) \), \( k(z, w) = \Phi \left( \frac{|z-w|^2}{Im(z)Im(w)} \right) \) for \( \Phi \in C_\infty(\mathbb{R}) \), and \( \Lambda(\lambda) \) is a entire function of \( \lambda \) (that depends solely on \( \Phi \)) that takes eigenvalues of \( \Delta \) to eigenvalues of the integral operator defined by \( (Lf)(z) = \int_{H} k(z, w) f(w) d\mu(w) \). [7] goes on to evaluate the integrals.
on the right-hand side of (3.4) in more detail, but we leave it in this form to compare it to Theorem 3.7. Note that (3.4) and (3.3) are almost identical in form (with integrals replaced by sums), except for two features: The function $\Lambda(\lambda)$ in (3.4) is replaced by $\Lambda(\{\lambda_{a}^{(i)}\}_{a \in F_{q}}) = \left( \sum_{a \in F_{q}} \Phi(a) \lambda_{a}^{(i)} \right)$ in Theorem 3.7, since there are $q$ operators in the finite case instead of just one as in the classical case. Also, in the finite case the terms $\frac{1}{|\text{Stab}(y)|}$ appear in the sum $I(\Phi, \gamma)$. This is because we are considering $\Gamma$ that may contain matrices with fixed points in $H_{q}$, while the trace formula in [7] considers strictly parabolic $\Gamma$.

**Remark 3.9.** One can view $H_{q}$ as a finite symmetric space $G/K$ as in [18, Chapter 19]. It can be shown that

$$L^{2}(K \setminus G/K) = \{ f : G \to \mathbb{C} \mid f(k_{1}gk_{2}) = f(g), \forall k_{1}, k_{2} \in K, g \in G \}$$

is a commutative algebra under convolution [18, pgs. 310, 322]. Let $f \in L^{2}(K \setminus G/K)$ and $g \in G$. Then,

$$\left( \delta_{S_{q}(\delta, a)} \ast f \right)(g) = (f \ast \delta_{S_{q}(\delta, a)})(g) = \sum_{h \in G} f(h) \delta_{S_{q}(\delta, a)}(h^{-1}g) = \sum_{h \in G \text{ and } h^{-1}g \in S_{q}(\delta, a)} f(h).$$

If $s \in S_{q}(\delta, a)$, then $a = d(s\sqrt{\delta}, \sqrt{\delta}) = d(\sqrt{\delta}, s^{-1}\sqrt{\delta}) = d(s^{-1}\sqrt{\delta}, \sqrt{\delta})$ implies that $s^{-1} \in S_{q}(\delta, a)$. Using this, along with the fact that $f \in L^{2}(K \setminus G/K)$, we have that

$$\left( \delta_{S_{q}(\delta, a)} \ast f \right)(g) = \sum_{h \in G \text{ and } g^{-1}h \in S_{q}(\delta, a)} f(h) = |K| \sum_{h \in \text{Aff}(q) \text{ and } g^{-1}h \in S_{q}(\delta, a)} f(h) = |K| \sum_{s \in S_{q}(\delta, a)} f(gs),$$

where $\text{Aff}(q)$ is given by (1.4). If we identify $L^{2}(K \setminus G/K)$ with $L^{2}(K \setminus H_{q})$ and $f(gs)$ with $f(gs\sqrt{\delta})$, then we have that $A_{a}^{K} f$ corresponds to $\frac{1}{|K|} \left( \delta_{S_{q}(\delta, a)} \ast f \right)$. One may consider the Hecke algebra $H(K, G)$ consisting of all formal finite linear combinations of $K$-double cosets in $G$ (see [9, pg. 6]). There is an isomorphism of algebras between $H(K, G)$ and $L^{2}(K \setminus G/K)$ that maps the double coset $S_{q}(\delta, a)$ to $\frac{1}{|K|} \delta_{S_{q}(\delta, a)}$ (see [9, pgs. 25–26]). Consider the action of $S_{q}(\delta, a)$ on $H(K, G)$ by left multiplication. In this context, $Tr(A_{a}^{K}) = Tr(S_{q}(\delta, a))$. 


Table 2. Congruency classes of $GL(2, \mathbb{F}_q)$.

<table>
<thead>
<tr>
<th>Representative</th>
<th># Elements in class</th>
<th># Classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(r \ 0)$, central</td>
<td>1</td>
<td>$q - 1$</td>
</tr>
<tr>
<td>$(r \ 1 \ 0 \ r)$, parabolic</td>
<td>$q^2 - 1$</td>
<td>$q - 1$</td>
</tr>
<tr>
<td>$(r \ 0 \ s \ 0), r \neq s$, hyperbolic</td>
<td>$q^2 + q$</td>
<td>$\frac{(q-1)(q-2)}{2}$</td>
</tr>
<tr>
<td>$(r \ s \ \delta \ r)$, $s \neq 0$, elliptic</td>
<td>$q^2 - q$</td>
<td>$\frac{q(q-1)}{2}$</td>
</tr>
</tbody>
</table>

4. Computing the trace formula for several $\Gamma$

In this section, we compute the trace formula for $\Gamma = \{I\}$, $N$, and $K$. In all our examples we take $n = |\Gamma \setminus H_q|$, and $\phi_1, \ldots, \phi_n$ to be a basis for $L^2(\Gamma \setminus H_q)$ with $A^\Gamma \phi_i(z) = \lambda_i \phi_i(z)$. Recall that $\Gamma \gamma = \{ \gamma' \in \Gamma | \gamma' \gamma = \gamma \gamma' \}$.

We will need Table 2 which gives the conjugacy classes of $GL(2, \mathbb{F}_q)$, where $q$ is odd and $\delta$ is a non-square in $\mathbb{F}_q$. It is taken from [18, pg. 366].

4.1 The trace formula for $\Gamma = \{I\}$

We start with an easy example to make sure our formula is working. If $\Gamma = \{(1 \ 0 \ 0 \ 1) = I\}$, then $\Gamma \setminus H_q = \Gamma \setminus H_q = H_q$, and $|\Gamma \setminus H_q| = q(q - 1)$. Since $Stab_\Gamma(z) = \{I\}$ for all $z \in H_q$, the trace formula (3.3) yields

$$Tr(L_\Phi) = I(\Phi, I) = \sum_{y \in H_q} k(Iy, y) = \Phi(0)q(q - 1) = |H_q| \Phi(0).$$

Letting $\Phi(a) = \delta_c(a)$ we have

$$Tr(A^\Gamma_c) = Tr(A_c) = \begin{cases} 0, & c \neq 0 \\ q(q - 1), & c = 0. \end{cases}$$

This is the trace of the adjacency operators $A_c$, which we already know to be zero, except, of course, for the case $c = 0$, which corresponds to the identity operator.

4.2 The trace formula for $\Gamma = N$

Let $\Gamma = N = \{(1 \ a) | a \in \mathbb{F}_q\}$. Then $|N \setminus H_q| = q - 1$ and a set of representatives for $N \setminus H_q$ is $\{a \sqrt{\delta} | a \neq 0\}$. In order to use the trace formula we need to know the conjugacy classes of $N$. Fortunately, no two elements of
$N$ are conjugate; thus, the sum over the conjugacy classes $\{\gamma\}$ of $N$ is just the sum over $\gamma \in N$. We break up the sum over $\gamma \in N$ into central and parabolic sums. Since $N$ is a commutative group, $N_\gamma = N$ and $N_\gamma \backslash H_q = N \backslash H_q$ for every $\gamma \in N$. Also, $Stab_N(z) = \{I\}$, for all $z \in H_q$.

First we compute the central terms of the sum. $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the only central term in our sum. The central term of (3.3) is

$$I(\Phi, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \sum_{N x \in N \backslash H_q} k(x, x) = \Phi(0) |N \backslash H_q| = \Phi(0)(q - 1).$$

Now we compute the parabolic terms. If $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b \in \mathbb{F}^*_q$, then

$$I(\Phi, \gamma) = \sum_{N x \in N \backslash H_q} k(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} x, x)$$

$$= \sum_{a \sqrt{\delta} \in N \backslash H_q} k(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} a \sqrt{\delta}, a \sqrt{\delta})$$

$$= \sum_{a \in \mathbb{F}^*_q} \Phi(d \left( b + a \sqrt{\delta}, a \sqrt{\delta} \right))$$

$$= \sum_{a \in \mathbb{F}^*_q} \Phi \left( \frac{N(b)}{Im(b + a \sqrt{\delta})Im(a \sqrt{\delta})} \right)$$

$$= \sum_{a \in \mathbb{F}^*_q} \Phi \left( \frac{b^2}{a^2} \right)$$

$$= \sum_{u \in \mathbb{F}^*_q} \Phi(u^2).$$

Putting the central and parabolic terms together, we have

$$Tr(L_\phi) = \Phi(0)(q - 1) + \sum_{b \in \mathbb{F}^*_q} \sum_{u \in \mathbb{F}^*_q} \Phi(u^2)$$

$$= \Phi(0)(q - 1) + (q - 1) \sum_{u \in \mathbb{F}^*_q} \Phi(u^2)$$

$$= (q - 1) \sum_{u \in \mathbb{F}^*_q} \Phi(u^2).$$

Therefore, the trace formula for $\Gamma = N$ is

$$\sum_{a \in \mathbb{F}^*_q} \Phi(a) Tr(A^N_a) = (q - 1) \sum_{u \in \mathbb{F}^*_q} \Phi(u^2). \quad (4.1)$$
Plug $\Phi(a) = \delta_c(a)$ into (4.1) to get that

$$Tr(A_N^c) = \begin{cases} (q - 1), & c = 0 \\ 2(q - 1), & c = \Box, c \neq 0 \\ 0, & c \neq \Box \end{cases}.$$ 

(In the formula above, we are using the notation $c = \Box$ to mean that $c$ is a square in $\mathbb{F}_q$.)

**Remark 4.1.** In the case of $\Gamma = N$, there is another way to find the trace of $A_N^c$. Given $a \in \mathbb{F}_q$, define the function $f_a$ on $x + y\sqrt{\delta} \in H_q$ to be

$$f_a(x + y\sqrt{\delta}) = \begin{cases} 1, & y = a \\ 0, & y \neq a \end{cases}.$$ 

The set of functions $\{f_a \mid a \in \mathbb{F}_q^*\}$ form a basis for the vector space $L^2(N\setminus H_q)$. We can find the diagonal entries of $A_N^c$ by computing the values of $(A_c f_a)(a\sqrt{\delta})$ for each $a \in \mathbb{F}_q$. We have the following:

$$\begin{align*}
(A_c f_a)(a\sqrt{\delta}) &= \sum_{\begin{smallmatrix} y \ x \\ 0 \ 1 \end{smallmatrix} \in S_q(\delta, c)} f_a \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \sqrt{\delta} \right) \\
 &= \sum_{\begin{smallmatrix} y \ x \\ 0 \ 1 \end{smallmatrix} \in S_q(\delta, c)} f_a(ax + ay\sqrt{\delta}) \\
 &= \sum_{\begin{smallmatrix} 1 \ x \\ 0 \ 1 \end{smallmatrix} \in S_q(\delta, c)} 1
\end{align*}$$

Recall that $\begin{smallmatrix} 1 \ x \\ 0 \ 1 \end{smallmatrix} \in S_q(\delta, c)$ if and only if $d(x + \sqrt{\delta}, \sqrt{\delta}) = c$ if and only if $x^2 = c$. If $c = 0$, then the diagonal entries of $A_N^c$ are all equal to 1. If $c \neq 0$ and $x^2 = c$ has a solution in $\mathbb{F}_q$, then the diagonal entries are all 2. If $c \neq 0$ and $x^2 = c$ does not have a solution in $\mathbb{F}_q$, then the diagonal entries are all 0. This yields the same result as the trace formula since $A_N^c$ has size $(q - 1) \times (q - 1)$.

4.3 The trace formula for $\Gamma = K$

Let

$$\Gamma = K = \{ \gamma \in G \mid \gamma \sqrt{\delta} = \sqrt{\delta} \} = \left\{ \begin{pmatrix} a & b \delta \\ b & a \end{pmatrix} \mid a, b \in \mathbb{F}_q, a^2 - \delta b^2 \neq 0 \right\}.$$
First notice that $K$ is a commutative group, and therefore, $K_{\gamma} = K$; second, note that

$$|Stab_K(y)| = \begin{cases} q - 1 & \text{if } y \neq \pm \sqrt{\delta} \\ q^2 - 1 & \text{if } y = \pm \sqrt{\delta} \end{cases}$$

(since $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ fixes all the elements of $H_q$ and $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right), b \neq 0$, only fixes $\pm \sqrt{\delta}$).

One can get a fundamental domain for $K \backslash H_q$ by picking an element of $H_q$ that is distance $a$ from $\sqrt{\delta}$ for each $a \in \mathbb{F}_q$ (see the note about the set of $K$ double cosets for $\text{GL}(2, \mathbb{F}_q)$ that is given after equation (1.8)). Recall that each of $\sqrt{\delta}$ and $-\sqrt{\delta}$ is its own $K$ orbit. We now compute the sums on the right side of the trace formula. We break the sum into sums over central and elliptic conjugacy classes.

The central terms are the matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ for $a \in \mathbb{F}_q^*$. The central terms contribute

$$\sum_{\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)} \sum_{K_{\gamma} \in K \backslash H_q} \frac{1}{|Stab_K(y)|} k(\gamma y, y) = \sum_{a \in \mathbb{F}_q^*} \sum_{K_{\gamma} \in K \backslash H_q} \frac{1}{|Stab_K(y)|} \Phi(0)$$

$$= (q - 1) \left[ \frac{2}{q^2 - 1} + \frac{q - 2}{q - 1} \right] \Phi(0)$$

$$= q \Phi(0) \left[ \frac{q - 1}{q + 1} \right]$$

$$= |K \backslash H_q| \Phi(0) \left[ \frac{q - 1}{q + 1} \right]$$

to the trace formula (3.3). The formula given above follows from the fact that there are $q - 2$ elements in $K \backslash H_q$ not equal to $K\sqrt{\delta}$ and $K(-\sqrt{\delta})$, and the comments mentioned above about $|Stab_K(y)|$.

The elliptic terms are the matrices of the form $\left(\begin{smallmatrix} a & b \delta \\ b & a \end{smallmatrix}\right), b \neq 0$. Note that

$$N(\left(\begin{smallmatrix} a & b \delta \\ b & a \end{smallmatrix}\right) z - z) = \frac{N(b(\delta - z^2))}{N(bz + a)} = \frac{b^2 N((\delta - z^2))}{N(bz + a)}$$

and (1.1) imply that

$$d(\left(\begin{smallmatrix} a & b \delta \\ b & a \end{smallmatrix}\right) z, z) = \frac{b^2 N(\delta - z^2)}{Im(gz)Im(z)N(bz + a)}$$

$$= \frac{b^2 N(\delta - z^2)N(bz + a)}{Im(z)^2(a^2 - b^2 \delta)N(bz + a)}$$

$$= \frac{N(\delta - z^2)}{((\frac{z}{z})^2 - \delta)Im(z)^2}.$$
Using the formula above, we see that the elliptic terms contribute

\[
\sum_{\gamma = \begin{pmatrix} a & b \\ b & a \end{pmatrix}} \sum_{K_y \in K \setminus H_q} \frac{1}{|Stab_K(y)|} k(\gamma y, y)
\]

\[
= \sum_{\gamma = \begin{pmatrix} a & b \\ b & a \end{pmatrix}} \sum_{K_y \in K \setminus H_q} \frac{1}{|Stab_K(y)|} \Phi \left( d \left( \begin{array}{c} ay + b \delta \\ by + a \delta \end{array} \right) \right)
\]

\[
= \sum_{a \in \mathbb{F}_q} \sum_{b \in \mathbb{F}_q^*} \sum_{K_y \in K \setminus H_q} \frac{1}{|Stab_K(y)|} \Phi \left( \frac{N(\delta - y^2)}{(u^2 - \delta) \text{Im}(y)^2} \right)
\]

\[
= (q - 1) \sum_{a \in \mathbb{F}_q} \sum_{K_y \in K \setminus H_q} \frac{1}{|Stab_K(y)|} \Phi \left( \frac{N(\delta - y^2)}{(u^2 - \delta) \text{Im}(y)^2} \right)
\]

to the trace formula (3.3). Remember that there is a bijection between $K \setminus H_q$ and $\mathbb{F}_q$, and so the following claim will be useful in evaluating the above sum.

**Claim 4.2.** If $d(y, \sqrt{\delta}) = c \in \mathbb{F}_q$, then $\frac{N(\delta - y^2)}{(u^2 - \delta) \text{Im}(y)^2} = \frac{c(c - 4\delta)}{u^2 - \delta}$.

**Proof.** The formulas

\[
N(\delta - y^2) = N(y - \sqrt{\delta}) N(y + \sqrt{\delta}) = c \text{Im}(y) N(y + \sqrt{\delta})
\]

\[
N(y + \sqrt{\delta}) = c \text{Im}(y) - 4\delta \text{Im}(y).
\]

yield

\[
\frac{N(\delta - y^2)}{(u^2 - \delta) \text{Im}(y)^2} = \frac{N(y - \sqrt{\delta}) N(y + \sqrt{\delta})}{(u^2 - \delta) \text{Im}(y)^2} = \frac{c \text{Im}(y) N(y + \sqrt{\delta})}{(u^2 - \delta) \text{Im}(y)^2}
\]

\[
= \frac{c N(y + \sqrt{\delta})}{(u^2 - \delta) \text{Im}(y)}
\]

\[
= \frac{c(\text{Im}(y) - 4\delta \text{Im}(y))}{(u^2 - \delta) \text{Im}(y)}
\]

\[
= \frac{c(c - 4\delta)}{u^2 - \delta}.
\]

\[\square\]
Therefore, the contribution of the elliptic terms to the trace formula is

\[
(q - 1) \sum_{u \in \mathbb{F}_q} \sum_{K_y \in K \setminus H_q \atop d(y, \sqrt{\delta}) = c} \frac{1}{|\text{Stab}_K(y)|} \Phi \left( \frac{c(c - 4\delta)}{(u^2 - \delta)} \right). \tag{4.2}
\]

To compute (4.2) we need a lemma.

**Lemma 4.3.** If \( a \in \mathbb{F}_q^* \) and \( N \) is the number of solutions to \( ax_1^2 - x_2^2 = a\delta - 4\delta^2 \) over \( \mathbb{F}_q \), then

\[
N = \begin{cases} 
1 & \text{if } a = 4\delta \\
q - 1 & \text{if } a \neq 4\delta, a = \square \\
q + 1 & \text{if } a \neq 4\delta, a \neq \square
\end{cases}
\tag{4.3}
\]

Recall that \( a = \square \) means that \( a \) is a square in \( \mathbb{F}_q \).

**Proof.** If \( a = 4\delta \), then there is only one solution since \( \delta \) is a non-square. Let \( \chi \) denote the quadratic character on \( \mathbb{F}_q \). Theorem 10.5 of [3] gives that \( N = q - \chi(a) \) if \( a \neq 4\delta \).

How are we going to use this result? Given \( a \in \mathbb{F}_q \), we are interested in the number of solutions to \( \frac{c(c - 4\delta)}{u^2 - \delta} = a \). This is the same as solving the equation \( au^2 - c^2 + 4\delta c = a\delta \). Making the substitution \( d = c - 2\delta \) we get the equation \( au^2 - d^2 = a\delta - 4\delta^2 \). When \( a = 0 \) we are unable to use Lemma 4.3, but it is easy to count the solutions in this case. We get \( 2q \) solutions corresponding to \( c = 0, 4\delta \) and \( u = \text{anything} \). Applying Lemma 4.3 and what we just said about \( a = 0 \) to (4.2), we have

\[
(q - 1) \left( \frac{2q}{q^2 - 1} \Phi(0) + \frac{1}{q - 1} \Phi(4\delta) + \sum_{a \in \mathbb{F}_q^* \atop a \neq \square} \frac{q + 1}{q - 1} \Phi(a) + \sum_{a \in \mathbb{F}_q^* \atop a \neq 4\delta} \frac{q - 1}{q - 1} \Phi(a) \right)
\]

for the contribution of the elliptic terms to the trace formula.

Summing the central and elliptic terms gives us the trace formula for \( \Gamma = K \):

\[
\sum_{a \in \mathbb{F}_q} \Phi(a) \text{Tr}(A^K_a) = q\Phi(0) + \Phi(4\delta) + \sum_{a \in \mathbb{F}_q^* \atop a \neq \square} (q + 1)\Phi(a) + \sum_{a \in \mathbb{F}_q^* \atop a \neq 4\delta} (q - 1)\Phi(a). \tag{4.4}
\]
Plug $\Phi(a) = \delta_c(a)$ into (4.4) to get that

$$Tr(A^K_a) = \begin{cases} 
q = |K \setminus H_q|, & c = 0 \\
1, & c = 4\delta \\
q + 1, & c \neq \Box, c \neq 4\delta \\
q - 1, & c = \Box, c \neq 0
\end{cases}$$

**Remark 4.4.** In Remark 4.1 we found that one can compute the trace of $A^N_a$ directly without the trace formula. The author does not know of a similar method of computing the trace of $A^K_a$.

In [12, pg. 108] and [1, pg. 20], the operators $A^K_a$ are studied under the guise of collapsed adjacency matrices. It can be shown that the entries $c_{i,j}$ of the collapsed adjacency matrix corresponding to $X_q(\delta, a)$ satisfy $c_{i,j} \leq 2$ except for $c_{0,a}$ and $c_{4\delta, 4\delta-a}$ (which do not lie on the diagonal of the matrix if $a \neq 0$). The proof, however, does not tell us how to determine which entries are equal to 0, 1, or 2 in general. (The proof constructs a quadratic polynomial using one arbitrary element of $S_q(\delta, a)$ and one arbitrary element of $S_q(\delta, b)$. It then uses the fact that a quadratic polynomial over $F_q$ can have at most 2 roots.) The problem is that there are no explicit formulas for the elements of $S_q(\delta, a)$.

We leave the reader with an example. If $q = 3^2$, $F_9 = \{a + bx \mid 1 + x^2 = 0, a, b \in F_3\}$, and $\delta = 1 + x$, then with respect to a standard basis for $L^2(K \setminus H_q)$ we have that

$$A^K_1 = \begin{pmatrix}
2 & 1 & 2 & 0 & 2 & 0 & 2 & 0 & 1 \\
1 & 0 & 2 & 2 & 1 & 2 & 2 & 0 & 0 \\
2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 0 \\
0 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\
2 & 1 & 0 & 2 & 2 & 2 & 0 & 1 & 0 \\
0 & 2 & 2 & 2 & 2 & 0 & 2 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and

$$A^K_{2+2x} = \begin{pmatrix}
1 & 0 & 2 & 2 & 1 & 2 & 2 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 0 \\
1 & 0 & 2 & 2 & 1 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
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