Recall: $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$

is continuous at $a \in D$ if for every $\varepsilon > 0$ $\exists\, \delta > 0$ where if $x \in D$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

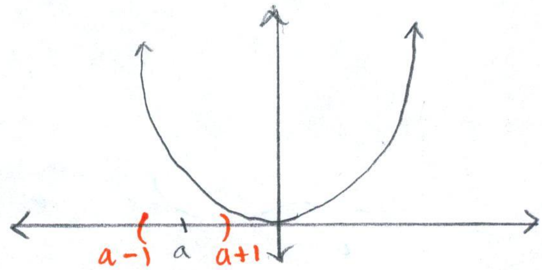
We will show that f is continuous on all of $\mathbb{R}$.

proof: Let $a \in \mathbb{R}$

We will show that $f(x) = x^2$ is continuous at a.

let $\varepsilon > 0$

Note that $|x^2 - a^2| = \underbrace{|x+a|}_{\text{bound with starting bound on } \delta} \underbrace{|x-a|}_{\text{bound this via } \delta}$



Start by assuming $\delta \le 1$. Then if

$\boxed{|x-a| < \delta \le 1}$ we have $|x+a| = |x-a+a+a| = |x-a+2a|$

$\le |x-a| + |2a| < 1 + 2|a|$

so if $|x-a| < 1$, then,

$$|x^2 - a^2| = |x+a||x-a| < (1+2|a|)|x-a|$$

let $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$

If $|x-a| < \delta$, then $|x^2-a^2| = |x+a||x-a| \underset{|x-a|<1}{<} (1+2|a|)|x-a|$

$\underset{|x-a| < \frac{\varepsilon}{1+2|a|}}{<} (1+2|a|)\dfrac{\varepsilon}{1+2|a|} = \varepsilon$ ☒

-Note, in this case we assume $\delta \le 1$, which works because $f(x) = x^2$ is defined in all $\mathbb{R}$.

However if $f(x) =$

Theorem: Let $D \subseteq \mathbb{R}$

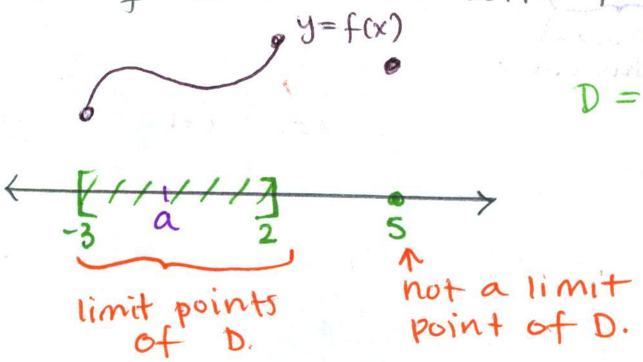
Let $a \in D$ and $\alpha \in \mathbb{R}$

Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$

If f and g are continuous at a then so are αf, $f+g$, $f-g$, and fg. If $f(a) \neq 0$ and f is continuous at a, then $\frac{1}{f}$ is continuous at a.

proof:

If a is not a limit point of D, then all the functions αf, $f+g$, $f-g$, fg, $\frac{1}{f}$ are continuous at a (with $\frac{1}{f}$ continuous with $f(a) \neq 0$)



$D = [-3, 2] \cup \{5\}$

f is continuous at 5

Suppose a is a limit point of D. Suppose f and g are continuous at a. Let's show fg is continuous at a

since a is a limit point of D and f and g are continuous at a, we get

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a)$$

By thms on limits,

$$\lim_{x \to a} (f(x)g(x)) \underset{\substack{\uparrow \\ \text{limit thm.}}}{=} \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = f(a) \cdot g(a)$$

So, fg is continuous at a. ▨

same idea for αf, $f \pm g$, $\frac{1}{f}$.