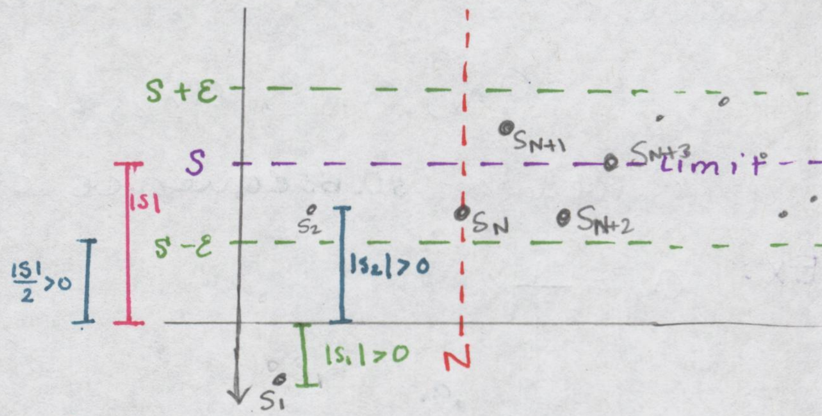
HW 2 #8

(a) Let (s_n) converge to $s \neq 0$. Assume $s_n \neq 0 \ \forall n$. Show that $\exists M > 0$ where $|s_n| \geq M \ \forall n$.



proof: Let $\varepsilon = \frac{|s|}{2} > 0$ ← since $s \neq 0$

Since $\lim_{n\to\infty} s_n = s$, $\exists N > 0$ where if $n \geq N$ then

$$|s_n - s| < \varepsilon$$

Note that if $n \geq N$ then,

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \underbrace{\frac{|s|}{2}}_{\varepsilon} + |s_n|$$

So, $|s| < \frac{|s|}{2} + |s_n|$

Thus, $\frac{|s|}{2} < |s_n| \ \forall n \geq N$.

Let $M = \min\{|s_1|, |s_2|, \dots, |s_{N-1}|, \frac{|s|}{2}\}$

Note that $M > 0$ since $s_n \neq 0$, $s \neq 0$

If $1 \leq n \leq N-1$, then $|s_n| \geq |s_n|$

If $N \leq n$, then $|s_n| \geq \frac{|s|}{2}$

So $\forall n$, $|s_n| \geq M$ $\boxtimes$

(b) If $s_n \neq 0 \ \forall n$, $s \neq 0$, and $\lim_{n\to\infty} s_n = s$, the $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

proof: Let $\varepsilon > 0$, By part (a) $\exists M > 0$ where $|s_n| > M \ \forall n$.

Note that $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s \cdot s_n}\right| = \frac{|s - s_n|}{|s||s_n|} < \frac{|s - s_n|}{|s| \cdot M} \ \forall n$

since $\lim_{n\to\infty} s_n = s$ $\exists N > 0$ where if $n \geq N$ then $|s - s_n| < \underbrace{\varepsilon \cdot |s| \cdot M}_{>0}$

So if $n \geq N$, then $\left|\frac{1}{s_n} - \frac{1}{s}\right| < \frac{|s - s_n|}{|s| \cdot M} < \frac{\varepsilon |s| \cdot M}{|s| \cdot M} = \varepsilon$ $\boxtimes$