~~More space for problems 7 and~~ 8

HW #5

1 (a)

S_3	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$
$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$
$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$
$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$
$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$
$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$
$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$

1 (b)

element	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$
order	1	2	2	2	3	3

1 (c)

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}^{-1}=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}^{-1}=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}^{-1}=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}^{-1}=\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$$

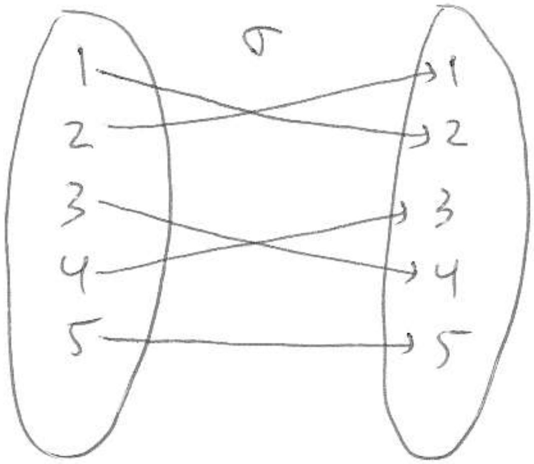
$$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}^{-1}=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}^{-1}=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$$

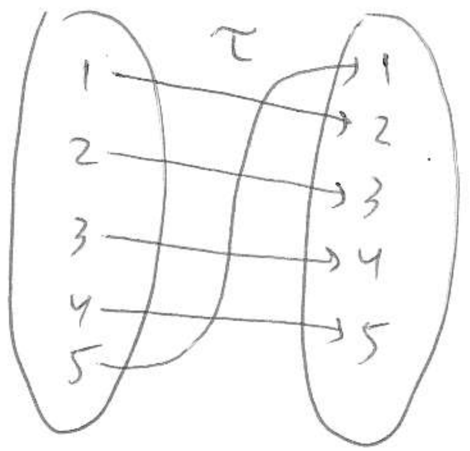
More space for problems 7 and 8

2

(a)



(b)



(c) $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix}$

(d) $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

(e) $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$

$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

(f) $\tau^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$

$\tau\sigma^2\tau^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

(g) From (d) we see that $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.
So, $\langle \sigma \rangle = \{ i, \sigma \}$ where i is the identity of S_5.

(h) $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$, $\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$, $\tau^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$,
$\tau^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$, $\tau^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.

Thus the order of τ is 5.

More space for problems 7 and 8

③

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 7 & 4 & 3 & 1 & 6 & 5 & 8 \end{pmatrix} = (1,2,7,5)(3,4)$

$= (1,5)(1,7)(1,2)(3,4)$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 5 & 2 & 7 & 8 & 6 \end{pmatrix} = (1,3,4,5,2)(6,7,8)$

$= (1,2)(1,5)(1,4)(1,3)(6,8)(6,7)$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 5 & 3 & 7 & 1 & 4 & 2 & 9 & 8 & 10 \end{pmatrix}$

$= (1,6,4,7,2,5)(8,9)$

$= (1,5)(1,2)(1,7)(1,4)(1,6)(8,9)$

More space for problems 7 and 8

(4)

(a) <u>φ_g is a homomorphism:</u> Let $x, y \in G$. Then

$$\varphi_g(xy) = g^{-1}xyg = g^{-1}xgg^{-1}yg = \varphi_g(x)\varphi_g(y).$$

<u>φ_g is 1-1:</u> Suppose $\varphi_g(x) = \varphi_g(y)$ for some $x, y \in G$.
Then $g^{-1}xg = g^{-1}yg$. So, $gg^{-1}xgg^{-1} = gg^{-1}ygg^{-1}$.
Thus, $x = y$.

<u>φ_g is onto:</u> Let $z \in G$. Then $gzg^{-1} \in G$ and

$$\varphi_g(gzg^{-1}) = g^{-1}(gzg^{-1})g = z.$$

(b) Note that the inverse of $\bar{3}$ is $\bar{2}$ in $\mathbb{Z}_5$. So, $\varphi_{\bar{3}}(x) = \bar{2} + x + \bar{3}$.

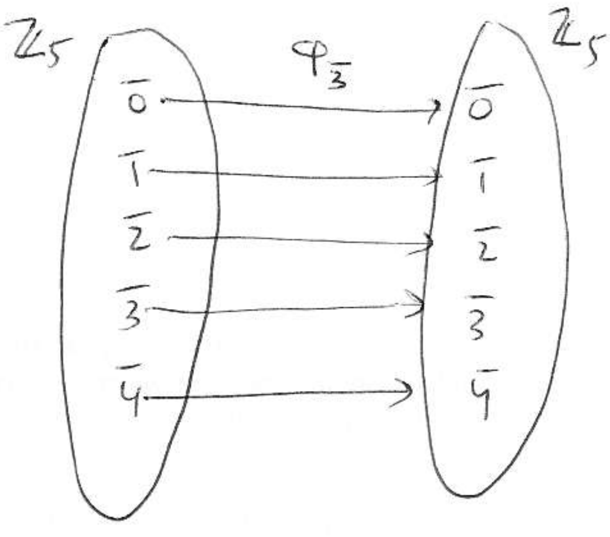
$\varphi_{\bar{3}}(\bar{0}) = \bar{2} + \bar{0} + \bar{3} = \bar{0}$

$\varphi_{\bar{3}}(\bar{1}) = \bar{2} + \bar{1} + \bar{3} = \bar{1}$

$\varphi_{\bar{3}}(\bar{2}) = \bar{2} + \bar{2} + \bar{3} = \bar{2}$

$\varphi_{\bar{3}}(\bar{3}) = \bar{2} + \bar{3} + \bar{3} = \bar{3}$

$\varphi_{\bar{4}}(\bar{3}) = \bar{2} + \bar{4} + \bar{3} = \bar{4}$



More space for problems 7 and 8

(c) $D_{10} = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$

The inverse of r is r^4. Hence $\varphi_r(x) = r^{-1}xr = r^4xr$.

$\varphi_r(1) = r^4 1 r = 1$

$\varphi_r(r) = r^4 r r = r$

$\varphi_r(r^2) = r^4 r^2 r = r^2$

$\varphi_r(r^3) = r^4 r^3 r = r^3$

$\varphi_r(r^4) = r^4 r^4 r = r^4$

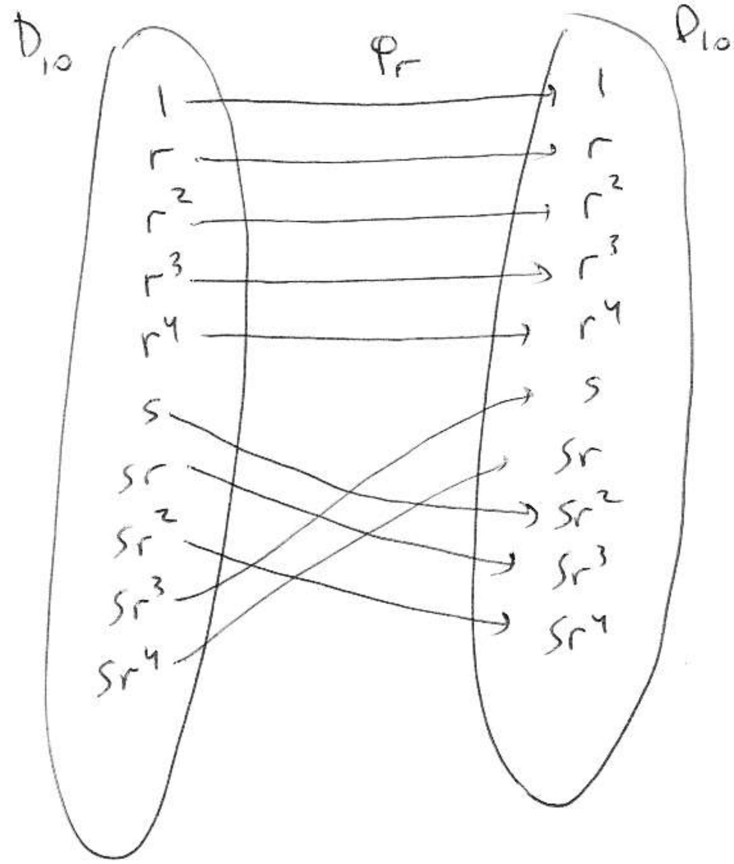
$\varphi_r(s) = r^4 s r = s r^{-4} r$
$= s r^{-3} = s r^2$

$\varphi_r(sr) = r^4 s r r = s r^{-4} r^2$
$= s r^{-2} = s r^3$

$\varphi_r(sr^2) = r^4 s r^2 r = s r^{-4+3}$
$= s r^{-1} = s r^4$

$\varphi_r(sr^3) = r^4 s r^3 r = s$

$\varphi_r(sr^4) = r^4 s r^4 r = sr$



(d) Note that $s^{-1} = s$. Hence $\varphi_s(x) = sxs$.

$\varphi_s(1) = s1s = 1$

$\varphi_s(r) = srs = r^{-1} = r^4$

$\varphi_s(r^2) = sr^2s = r^{-2} = r^3$

$\varphi_s(r^3) = sr^3s = r^{-3} = r^2$

$\varphi_s(r^4) = sr^4s = r^{-4} = r$

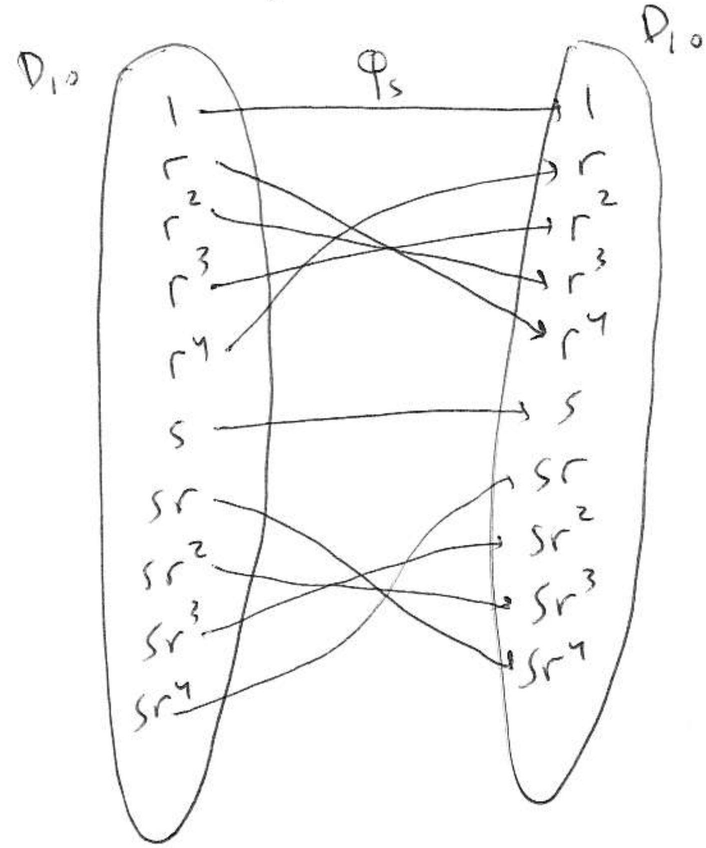
$\varphi_s(s) = sss = s$

$\varphi_s(sr) = ssrs = sr^{-1} = sr^4$

$\varphi_s(sr^2) = ssr^2s = sr^{-2} = sr^3$

$\varphi_s(sr^3) = ssr^3s = sr^{-3} = sr^2$

$\varphi_s(sr^4) = ssr^4s = sr^{-4} = sr$



---

(5) $\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$\varphi_{\bar{0}} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{pmatrix}$

$\varphi_{\bar{1}}(x) = \bar{1} + x \Rightarrow \varphi_{\bar{1}} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{0} \end{pmatrix}$

$\varphi_{\bar{2}}(x) = \bar{2} + x \Rightarrow \varphi_{\bar{2}} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{2} & \bar{3} & \bar{4} & \bar{0} & \bar{1} \end{pmatrix}$

$\varphi_{\bar{3}}(x) = \bar{3} + x \Rightarrow \varphi_{\bar{3}} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{3} & \bar{4} & \bar{0} & \bar{1} & \bar{2} \end{pmatrix}$

$\varphi_{\bar{4}}(x) = \bar{4} + x \Rightarrow \varphi_{\bar{4}} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{4} & \bar{0} & \bar{1} & \bar{2} & \bar{3} \end{pmatrix}$

Let $H = \left\{ \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{2} & \bar{3} & \bar{4} & \bar{0} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{3} & \bar{4} & \bar{0} & \bar{1} & \bar{2} \end{pmatrix}, \right.$
$\left. \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{4} & \bar{0} & \bar{1} & \bar{2} & \bar{3} \end{pmatrix} \right\}$. Then $H \cong \mathbb{Z}_5$.

(6) $D_4 = \{1, r, s, sr\}$

$\varphi_1(x) = 1x = x \Rightarrow \varphi_1 = \begin{pmatrix} 1 & r & s & sr \\ 1 & r & s & sr \end{pmatrix}$

$\varphi_r(x) = rx \Rightarrow \left\{ \begin{array}{l} \varphi_r(1) = r \cdot 1 = r \\ \varphi_r(r) = rr = r^2 = 1 \\ \varphi_r(s) = rs = sr^{-1} = sr \\ \varphi_r(sr) = rsr = s \end{array} \right\} \Rightarrow \varphi_r = \begin{pmatrix} 1 & r & s & sr \\ r & 1 & sr & s \end{pmatrix}$

$\varphi_s(x) = sx \Rightarrow \left\{ \begin{array}{l} \varphi_s(1) = s \cdot 1 = s \\ \varphi_s(r) = sr \\ \varphi_s(s) = ss = 1 \\ \varphi_s(sr) = ssr = r \end{array} \right\} \Rightarrow \varphi_s = \begin{pmatrix} 1 & r & s & sr \\ s & sr & 1 & r \end{pmatrix}$

$\varphi_{sr}(x) = srx \Rightarrow \left\{ \begin{array}{l} \varphi_{sr}(1) = sr \cdot 1 = sr \\ \varphi_{sr}(r) = srr = s \\ \varphi_{sr}(s) = srs = r^{-1} = r \\ \varphi_{sr}(sr) = srsr = 1 \end{array} \right\} \Rightarrow \varphi_{sr} = \begin{pmatrix} 1 & r & s & sr \\ sr & s & r & 1 \end{pmatrix}$

$H = \left\{ \begin{pmatrix} 1 & r & s & sr \\ 1 & r & s & sr \end{pmatrix}, \begin{pmatrix} 1 & r & s & sr \\ r & 1 & sr & s \end{pmatrix}, \begin{pmatrix} 1 & r & s & sr \\ s & sr & 1 & r \end{pmatrix}, \begin{pmatrix} 1 & r & s & sr \\ sr & s & r & 1 \end{pmatrix} \right\}$

Then $H \leq S_{D_4}$ and $D_4 \cong H$.

---

(7) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 2 & 1 & 3 & 4 & 5 & \cdots & n \end{pmatrix} = (1,2)$

and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 3 & 2 & 4 & 5 & \cdots & n \end{pmatrix} = (2,3)$

Then $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 2 & 3 & 1 & 4 & 5 & \cdots & n \end{pmatrix} = (1,2,3)$

and $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 3 & 1 & 2 & 4 & 5 & \cdots & n \end{pmatrix} = (1,3,2)$

So, $\sigma\tau \neq \tau\sigma$.