<u>Claim:</u> Let G be a group and $x \in G$.

(a) If x has finite order n, then $\langle x \rangle = \{e, x, x^2, \dots, x^{n-1}\}$. Moreover, $x^k \neq x^h$ if $0 \leq k < h < n$. Hence, the order of x is $|\langle x \rangle|$.

(b) If x has infinite order, then $\langle x \rangle = \{\dots, x^{-2}, x^{-1}, e, x, x^2, \dots\}$ and $x^h \neq x^k$ if $h \neq k$.

<u>proof:</u> (a) Let $S = \{e, x, x^2, \dots, x^{n-1}\}$ and $m \in \mathbb{Z}$. We show that $x^m \in S$. We know that $m = qn + r$ where $q, r \in \mathbb{Z}$ and $0 \leq r < n$. Hence, $x^m = x^{qn+r} = (x^n)^q x^r = e \cdot x^r = x^r \in S$. Suppose now that $x^k = x^h$ with $0 \leq k < h < n$. Then $x^{h-k} = e$ with $0 < h - k < n$. This contradicts the fact that the order of x is n.

(b) Suppose $x^h = x^k$ with $h > k$. Then, $x^{h-k} = e$. This is a contradiction. ▨