Circular Nim Games

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Circular Nim CN\((n, k)\)

- \(n\) stacks of tokens arranged in a circle
- Select \(k\) consecutive stacks and remove at least one token from at least one of the stacks
- Last player to move wins

\(k = 1\) corresponds to regular Nim
Circular Nim CN\((n, k)\)

**Question**: For a given position, can we determine whether Player I or Player II has a winning strategy, that is, can make moves in such a way that s/he will win, no matter how the other player plays?

We will determine the set of losing positions, that is, all positions that result in a loss for the player playing from that position.
Combinatorial Games

Definition

An *impartial combinatorial game* has the following properties:

- each player has the **same moves** available at each point in the game (as opposed to chess, where there are white and black pieces).
- no randomness (dice, spinners) is involved, that is, each player has **complete information** about the game and the potential moves.
Analyzing $\text{CN}(n, k)$

**Definition**

A *position* in $\text{CN}(n, k)$ is denoted by $p = (p_1, p_2, \ldots, p_n)$, where $p_i \geq 0$ denotes the number of tokens in stack $i$. A position that arises from a move in the current position is called an *option*. The directed graph which has the positions as the nodes and an arrow between a position and its options is called the *game tree*.

We do not distinguish between a position and any of its rotations or reversals.
Options of position \((0, 1, 2)\) in \(\text{CN}(3, 2)\)

\[
\begin{align*}
(0, 1, 2) & \leadsto \\
(0, 1, 2) & \leadsto \\
(0, 1, 2) & \leadsto 
\end{align*}
\]
Options of position \((0, 1, 2)\) in \(\text{CN}(3, 2)\)

\[
(0, 1, 2) \mapsto (0, 0, 2) \\
(0, 1, 2) \mapsto \\
(0, 1, 2) \mapsto 
\]
Options of position \((0, 1, 2)\) in \(\text{CN}(3, 2)\)

\[
\begin{align*}
(0, 1, 2) & \leadsto (0, 0, 2) \\
(0, 1, 2) & \leadsto (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 1), (0, 1, 0) \\
(0, 1, 2) & \leadsto
\end{align*}
\]
Options of position \((0,1,2)\) in \(CN(3,2)\)

\[
(0, 1, 2) \sim (0, 0, 2)
\]

\[
(0, 1, 2) \sim (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 1), (0, 1, 0)
\]

\[
(0, 1, 2) \sim (0, 1, 1), (0, 1, 0)
\]
Options of position $(0, 1, 2)$ in $\text{CN}(3, 2)$

\begin{align*}
(0, 1, 2) & \sim (0, 0, 2) \\
(0, 1, 2) & \sim (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 1), (0, 1, 0) \\
(0, 1, 2) & \sim (0, 1, 1), (0, 1, 0) \\
\text{Overall} & \\
(0, 1, 2) & \sim (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 1)
\end{align*}
Game tree for CN(3, 2) position (0, 1, 2)
Impartial Games

Definition

A position is a \textit{P position} for the player about to make a move if the previous player can force a win (that is, the player about to make a move is in a losing position). The position is a \textit{N position} if the next player (the player about to make a move) can force a win.

For impartial games, there are only two outcome classes for any position, namely \textbf{winning position} (\textit{N} position) or \textbf{losing position} (\textit{P} position). The set of \textit{losing positions} is denoted by \textit{L}.
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree recursively as follows:

- **Leaves** of the game tree are always losing (\(P\)) positions.
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree recursively as follows:

- **Leaves** of the game tree are always losing ($\mathcal{P}$) positions.

Next we select any position (node) whose options (offsprings) are all labeled. There are two cases:
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree \textit{recursively} as follows:

- \textbf{Leaves} of the game tree are always \textit{losing} ($\mathcal{P}$) positions.

Next we select any position (node) whose \textit{options} (offsprings) are all \textit{labeled}. There are two cases:

- The position has at least one option that is a losing ($\mathcal{P}$) position
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree recursively as follows:

- **Leaves** of the game tree are always losing (P) positions.

Next we select any position (node) whose options (offsprings) are all labeled. There are two cases:

- The position has at least one option that is a losing (P) position
- All options of the position are winning (N) positions
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree \textbf{recursively} as follows:

\begin{itemize}
  \item \textbf{Leaves} of the game tree are always \textit{losing} (\(P\)) positions.
  \item The position has at least one option that is a losing (\(P\)) position \(\Rightarrow\) \textbf{winning} position and should be labeled \(N\)
  \item All options of the position are winning (\(N\)) positions
\end{itemize}
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree recursively as follows:

- **Leaves** of the game tree are always losing ($\mathcal{P}$) positions.

Next we select any position (node) whose options (offsprings) are all labeled. There are two cases:

- The position has at least one option that is a losing ($\mathcal{P}$) position
  \[ \Rightarrow \text{winning position and should be labeled } \mathcal{N} \]

- All options of the position are winning ($\mathcal{N}$) positions
  \[ \Rightarrow \text{losing position and should be labeled } \mathcal{P} \]
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree recursively as follows:

- **Leaves** of the game tree are always losing \( (\mathcal{P}) \) positions.

Next we select any position (node) whose options (offsprings) are all labeled. There are two cases:

- The position has at least one option that is a losing \( (\mathcal{P}) \) position
  \[ \Rightarrow \text{winning} \] position and should be labeled \( \mathcal{N} \)

- All options of the position are winning \( (\mathcal{N}) \) positions
  \[ \Rightarrow \text{losing} \] position and should be labeled \( \mathcal{P} \)

The label of the starting position of the game then tells whether Player I \( (\mathcal{N}) \) or Player II \( (\mathcal{P}) \) has a winning strategy.
Labeling the game tree for CN(3, 2) position (0, 1, 2)
Labeling the game tree for $CN(3, 2)$ position $(0, 1, 2)$
Labeling the game tree for CN(3, 2) position (0, 1, 2)
Labeling the game tree for CN(3, 2) position (0, 1, 2)
Labeling the game tree for CN(3, 2) position (0, 1, 2)
An important tool

Theorem

Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets $A$ and $B$ with the properties:

I. every option of a position in $A$ is in $B$;

II. every position in $B$ has at least one option in $A$; and

III. the final positions are in $A$.

Then $A = \mathcal{L}$ and $B = \mathcal{W}$. 
Proof strategy

- Obtain a candidate set $S$ for the set of losing positions $\mathcal{L}$
- Show that any move from a position $p \in S$ leads to a position $p' \notin S$ (I)
- Show that for every position $p \notin S$, there is a move that leads to a position $p' \in S$ (II)

Note that the only final position is $(0, 0, \ldots, 0)$, and it is easy to see that (III) is satisfied in all cases.
Digital sum

Definition

The digital sum $a \oplus b \oplus \cdots \oplus k$ of of integers $a, b, \ldots, k$ is obtained by translating the values into their binary representation and then adding them without carry-over.

Note that $a \oplus a = 0$.

Example

The digital sum $12 \oplus 13 \oplus 7$ equals 6:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<tbody>
<tr>
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<td></td>
<td>0</td>
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The easy cases

Theorem

(1) *The game* $\text{CN}(n, 1)$ *reduces to* Nim, *for which the set of losing positions is given by* 
\[ \mathcal{L} = \{(p_1, p_2, \ldots, p_n) \mid p_1 \oplus p_2 \oplus \cdots \oplus p_n = 0\} \].

(2) *The game* $\text{CN}(n, n)$ *has a single losing position, namely* 
\[ \mathcal{L} = \{(0, 0, \ldots, 0)\} \].

(3) *The game* $\text{CN}(n, n - 1)$ *has losing positions* 
\[ \mathcal{L} = \]
Introduction
Tools from Combinatorial Game Theory
First Results
General results
n=4

The easy cases

Theorem

(1) The game $\text{CN}(n, 1)$ reduces to Nim, for which the set of losing positions is given by

$$\mathcal{L} = \{ (p_1, p_2, \ldots, p_n) \mid p_1 \oplus p_2 \oplus \cdots \oplus p_n = 0 \}.$$ 

(2) The game $\text{CN}(n, n)$ has a single losing position, namely

$$\mathcal{L} = \{ (0, 0, \ldots, 0) \}.$$ 

(3) The game $\text{CN}(n, n - 1)$ has losing positions

$$\mathcal{L} = \{ (a, a, \ldots, a) \mid a \geq 0 \}.$$
The easy cases

Theorem

(1) *The game* $CN(n, 1)$ *reduces to Nim, for which the set of losing positions is given by*

$$\mathcal{L} = \{(p_1, p_2, \ldots, p_n) \mid p_1 \oplus p_2 \oplus \cdots \oplus p_n = 0\}.$$  

(2) *The game* $CN(n, n)$ *has a single losing position, namely*

$$\mathcal{L} = \{(0, 0, \ldots, 0)\}.$$  

(3) *The game* $CN(n, n - 1)$ *has losing positions*

$$\mathcal{L} = \{(a, a, \ldots, a) \mid a \geq 0\}.$$  

This covers the games for $n = 1, 2, 3$. For $n = 4$, the only one game to consider is $CN(4, 2)$.  

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Circular Nim Games
Result for CN(4, 2)

Theorem

For the game CN(4, 2), the set of losing positions is
\[ \mathcal{L} = \{(a, b, a, b) \mid a, b \geq 0\}. \]
Result for CN(4, 2)

Theorem

For the game CN(4, 2), the set of losing positions is
\[ \mathcal{L} = \{(a, b, a, b) \mid a, b \geq 0\}. \]

Proof.

Let \( S = \{(a, b, a, b)\} \) and \( p \in S \). Playing on any stack results in a different value in its diagonal opposite stack \( \Rightarrow p' \notin S \).
**Result for CN(4, 2)**

**Theorem**

*For the game CN(4, 2), the set of losing positions is*

\[ \mathcal{L} = \{(a, b, a, b) \mid a, b \geq 0\}. \]

**Proof.**

Let \( S = \{(a, b, a, b)\} \) and \( p \in S \). Playing on any stack results in a different value in its diagonal opposite stack \( \Rightarrow p' \notin S \).

If \( p \notin S \):

```
4 5
2 3
```
Result for CN(4, 2)

Theorem

For the game CN(4, 2), the set of losing positions is \( \mathcal{L} = \{(a, b, a, b) \mid a, b \geq 0\} \).

Proof.

Let \( S = \{(a, b, a, b)\} \) and \( p \in S \). Playing on any stack results in a different value in its diagonal opposite stack \( \Rightarrow p' \notin S \).

If \( p \notin S \):

\[
\begin{array}{ccc}
4 & -1 & 5 - 3 \\
2 & & 3 \\
\end{array}
\]
Result for CN(4, 2)

**Theorem**

For the game CN(4, 2), the set of losing positions is  
\[ \mathcal{L} = \{(a, b, a, b) \mid a, b \geq 0\} \]

**Proof.**

Let \( S = \{(a, b, a, b)\} \) and \( p \in S \). Playing on any stack results in a different value in its diagonal opposite stack \( \Rightarrow p' \notin S \).

If \( p \notin S \):

\[
\begin{array}{cccc}
4 & 1 & 5 & 3 \\
2 & 3 & 2 & 3 \\
\end{array}
\]

\[
\rightarrow
\]

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Theorem (Dufour; Ehrenborg & Steingrímsson)

The game $\text{CN}(5, 2)$ has losing positions

$$L = \{(a^*, b, c, d, b) \mid a^* + b = c + d, a^* = \max(p)\}.$$ 

Note that $b$ has to be $\min(p)$. 

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Circular Nim Games
Result for CN(5, 2)

To show part (II), we can assume that \( \min(p) = 0 \). Two cases:

(i) \( \max(p) = w^* \) and \( \min(p) \) adjacent, \( p = (0, w^*, x, y, z) \)

\[ w^* \geq z + y: \]

\[ w^* < z + y: \]
Result for CN(5, 2)

(ii) max(p) and min(p) separated by one stack, p = (0, x + y, w, z, y), max(p) ∈ \{w, z\}

\[ z \geq x:\]

\[ z < x:\]
Theorem (Ehrenborg & Steingrímsson)

The game CN(5, 3) has losing positions
\[ \mathcal{L} = \{(0, b, c, d, b) \mid b = c + d\} \].

Note that \( b \) has to be \( \max(p) \). Proof similar to CN(5, 2) with more cases to be considered.
The big question

How do we find $\mathcal{L}$ ????
**Definition**

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by \( \text{mex}\{a, b, c, \ldots, k\} \).

**Example**

\[
\text{mex}\{1, 4, 5, 7\} = \\
\text{mex}\{0, 1, 2, 6\} = 
\]
Mex

Definition

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by $\text{mex}\{a, b, c, \ldots, k\}$.

Example

$$\text{mex}\{1, 4, 5, 7\} = 0$$
$$\text{mex}\{0, 1, 2, 6\} = \ldots$$
**Mex**

**Definition**

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by mex\{a, b, c, \ldots, k\}.

**Example**

\[
\begin{align*}
\text{mex}\{1, 4, 5, 7\} &= 0 \\
\text{mex}\{0, 1, 2, 6\} &= 3
\end{align*}
\]
The Grundy Function

Definition

The Grundy function $G(p)$ of a position $p$ is defined recursively as follows:

- $G(p) = 0$ for any final position $p$.
- $G(p) = \text{mex}\{G(q) | q \text{ is an option of } p\}$. 

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Circular Nim Games
The Grundy Function

Definition

The Grundy function $G(p)$ of a position $p$ is defined recursively as follows:

- $G(p) = 0$ for any final position $p$.
- $G(p) = \text{mex}\{G(q)|q \text{ is an option of } p\}$.

Theorem

*For a finite impartial game, $p$ belongs to class $P$ if and only if $G(p) = 0$.***
Recursive computation of Grundy function

(0, 1, 2)

(0, 0, 2)

(0, 0, 1)

(0, 0, 0)

(0, 1, 1)

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Circular Nim Games
Finding candidate set for $\mathcal{L}$

- Write program that computes options for a given position and then recursively computes Grundy function for each position
- Filter out those positions that have Grundy value zero
- **CREATIVITY** - find pattern
- Write program that computes values to check your pattern
- If pattern holds for large enough number of examples, try to prove it!
Result for CN(6, 3)

Theorem

For the game CN(6, 3), the set of losing positions is given by
\[ \mathcal{L} = \{(a, b, c, d, e, f)|a + b = d + e, b + c = e + f\}. \]

Note that also \( c + d = f + a \).
Result for CN(6, 4)

Theorem

For the game CN(6, 4), the set of losing positions is given by
\[ L = \{(a, b, c, d, e, f) | a + b = d + e, b + c = e + f, a \oplus c \oplus e = 0, a = \min(p)\} \].

Note that also \( c + d = f + a \).
Proof of $\mathcal{L}_{\text{CN}(6,4)}$ uses two lemmas:

**Lemma**

*If the position $p = (a, b, c, d, e, f) \in \mathcal{L}_{\text{CN}(6,4)}$ has a minimal value in each of the two triples $(a, c, d)$ and $(b, d, f)$, then $p = (a, b, c, a, b, c)$.***

**Lemma**

*For any set of positive integers $x_1, x_2, \ldots, x_n$ there exists an index $i$ and a value $x'_i$ such that $0 \leq x'_i \leq x_i$ and

$$x_1 \oplus \cdots \oplus x_{i-1} \oplus x'_i \oplus x_{i+1} \oplus \cdots \oplus x_n = 0.$$*
Result for $\text{CN}(6, 2)$

- Difficult case to prove - we need \textbf{ALL} Grundy values for a special substructure
- Same substructure occurs in all $\text{CN}(n, 2)$ games for $n \geq 6$
- Structure also occurs in other games such as $\text{CN}(9, 3)$
Conjecture for $\text{CN}(2m, m)$

$$\mathcal{L}_{\text{CN}(4,2)} = \{(a, b, c, d) \mid a + b = c + d \land b + c = a + d\}$$

$$\mathcal{L}_{\text{CN}(6,3)} = \{(a, b, c, d, e, f) \mid a + b = d + e \land b + c = e + f\}$$
Conjecture for CN(2m, m)

\[
\mathcal{L}_{\text{CN}(4,2)} = \{(a, b, c, d) \mid a + b = c + d \land b + c = a + d\}
\]

\[
\mathcal{L}_{\text{CN}(6,3)} = \{(a, b, c, d, e, f) \mid a + b = d + e \land b + c = e + f\}
\]

**Conjecture:**
Sums of pairs that are diagonally across are the same
Conjecture for $\text{CN}(2m, m)$

$$\mathcal{L}_{\text{CN}(4,2)} = \{(a, b, c, d) \mid a + b = c + d \land b + c = a + d\}$$

$$\mathcal{L}_{\text{CN}(6,3)} = \{(a, b, c, d, e, f) \mid a + b = d + e \land b + c = e + f\}$$

**Conjecture:**
Sums of pairs that are diagonally across are the same  NO
We have some partial results/conjectures for $n = 7, 8, 9$.

Specifically, $\mathcal{L}_{\text{CN}(8,6)} = \{(0, x, a, b, e, c, d, x) \mid a + b = c + d = x, e = \min\{x, a + d\}\}$. 
Example for $\text{CN}(8, 6)$

Can you find a move that results in a losing position?
Example for CN(8, 6)

Can you find a move that results in a losing position?

\[
\begin{array}{cccccccc}
5 & & 4 & & 6 & & 2 & & 4 & & 6 \\
7 & 3 & & 6 & & 2 & & 6 & & 2 \\
5 & 4 & & 0 & & 6 & & 4 \\
8 & & & & & 6 & & & & \\
\end{array}
\]
Example for CN(8, 6)

Can you find a move that results in a losing position?

\[
\begin{array}{cccc}
5 & 4 & 6 & 2 \\
7 & 3 & 2 & 1 \\
5 & 4 & 4 & 0 \\
8 & & 4 & \\
\end{array}
\]
Variations of Circular Nim

- Select a fixed number $a$ from at least one of the stacks
Variations of Circular Nim

- Select a fixed number $a$ from at least one of the stacks
- Select a fixed number $a$ from each of the heaps
Variations of Circular Nim

- Select a fixed number $a$ from at least one of the stacks
- Select a fixed number $a$ from each of the heaps
- Select at least one token from each of the $k$ heaps
- Select at least $a$ tokens from each of the $k$ heaps
Variations of Circular Nim

- Select a fixed number $a$ from at least one of the stacks
- Select a fixed number $a$ from each of the heaps
- Select at least one token from each of the $k$ heaps
- Select at least $a$ tokens from each of the $k$ heaps

Note that there is a different dynamic when the requirement is to select from each stack, as a zero stack now splits the position into separate positions with smaller $n$ and symmetries disappear.
Variations of Circular Nim

- Select a fixed number $a$ from at least one of the stacks
- Select a fixed number $a$ from each of the heaps
- Select at least one token from each of the $k$ heaps
- Select at least $a$ tokens from each of the $k$ heaps
- Select a total of at least $a$ tokens from the $k$ stacks

Note that there is a different dynamic when the requirement is to select from each stack, as a zero stack now splits the position into separate positions with smaller $n$ and symmetries disappear.
Variations of Circular Nim

- Select a fixed number $a$ from at least one of the stacks
- Select a fixed number $a$ from each of the heaps
- Select at least one token from each of the $k$ heaps
- Select at least $a$ tokens from each of the $k$ heaps
- Select a total of at least $a$ tokens from the $k$ stacks
- Select a total of exactly $a$ tokens from the $k$ stacks

Note that there is a different dynamic when the requirement is to select from each stack, as a zero stack now splits the position into separate positions with smaller $n$ and symmetries disappear.
Variations of Circular Nim

- Select a fixed number \( a \) from at least one of the stacks
- Select a fixed number \( a \) from each of the heaps
- Select at least one token from each of the \( k \) heaps
- Select at least \( a \) tokens from each of the \( k \) heaps
- Select a total of at least \( a \) tokens from the \( k \) stacks
- Select a total of exactly \( a \) tokens from the \( k \) stacks
- ... 

Note that there is a different dynamic when the requirement is to select from each stack, as a zero stack now splits the position into separate positions with smaller \( n \) and symmetries disappear.
Thank You!
References and Further Reading

*Winning Ways for Your Mathematical Plays, Vol 1 & 2.*  

*Lessons in Play.*  

R. Ehrenborg and E. Steingrímsson.  
Playing Nim on a simplicial complex.  