Compositions and Multisets Restricted by Patterns of Length 3

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joint work with

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Pattern Avoidance

- Knuth - Permutations avoiding a permutation pattern of length 3 $\rightarrow$ Catalan numbers
- Simion & Schmidt - Permutations avoiding any given set of permutation patterns of length 3
- Burstein - Words avoiding a set of permutation patterns of length 3
- Burstein & Mansour - Words avoiding patterns with repeated letters
- Heubach & Mansour - Compositions avoiding patterns of length 2
- Savage & Wilf - Compositions avoiding permutation patterns of length 3
Things to come ...

- Compositions and multisets avoiding a single pattern of length 3 with repeated letters

- Compositions and multisets avoiding pairs of patterns of length 3 with repeated letters

- Compositions and multisets avoiding special patterns of arbitrary length: 111...111 & 11...121...11
Notation and Definitions

- \( \mathbb{N} \) = set of all positive integers
- \( A = \{a_1, a_2, \ldots, a_d\} \) ordered subset of \( \mathbb{N} \); \( A' = A - \{a_1\}, \tilde{A} = A - \{a_d\} \)
- \([k] = \{1, 2, \ldots, k\}; [k]^n = \) set of all words of length \( n \) over \([k]\)
- pattern \( \tau = \) word in \([\ell]^k\) that contains each letter from \([\ell]\), possible with repetitions. Sets of patterns denoted by \( T \)
- \( S = 1^{m_1}2^{m_2}\ldots k^{m_k}, m_i > 0 \) multiset
- \( \mathfrak{S}_{m_1m_2\ldots m_k} = \) set of permutations on a multiset \( S \)
- \( \sigma = \sigma_1\sigma_2\ldots\sigma_m = \) composition of \( n \in \mathbb{N} \) with \( m \) parts where \( \sum_{i=1}^{m} \sigma_i = n \)
- \( C_n^A (C_{n;m}^A) = \) the set of all compositions of \( n \) with parts in \( A \) (\( m \) parts in \( A \))
Notation and Definitions

- \( \sigma \in C_n^A (C_{n;m}^A) \) contains \( \tau \) if \( \sigma \) contains a subsequence isomorphic to \( \tau \). Otherwise, \( \sigma \) avoids \( \tau \) and we write \( \sigma \in C_n^A (\tau) \ (\sigma \in C_{n;m}^A (\tau)) \)

- \( T_1 \) and \( T_2 \) are Wilf-equivalent, denoted by \( T_1 \mathrel{\text{Wilf}} \equiv T_2 \), if \( |C_{n;m}^A (T_1)| = |C_{n;m}^A (T_2)| \) for all \( A, m \) and \( n \).

- reversal map \( r(\tau) = r(\tau_1 \tau_2 \ldots \tau_k) = \tau_k \tau_{k-1} \ldots \tau_1 \); \( r(\tau) \mathrel{\text{Wilf}} \equiv \tau \)

- \( \{\tau, r(\tau)\} = \text{symmetry class of } \tau \)

- Generating functions

  - \( C_T^A (x; m) = \sum_{n \geq 0} |C_{n;m}^A (T)| x^n \)
  - \( C_T^A (x, y) = \sum_{m \geq 0} C_T^A (x; m) y^m \)
  - \( C_T^A (x) = C_T^A (x, 1) = \sum_{n \geq 0} |C_n^A (T)| x^n \).
Single Patterns of Length 3

- 111
- 121 and 112
- 221 and 212
- 123, 132 and 213 (permutation patterns)
  Wilf equivalence for compositions and multisets and generating function for compositions (Savage and Wilf)
The pattern 111

**Theorem:** Let $A = \{a_1, \ldots, a_d\}$ be any ordered finite or infinite set of positive integers. Then

$$
\sum_{m \geq 0} C^A_{111}(x; m) \frac{y^m}{m!} = \prod_{a \in A} \left(1 + x^a y + \frac{1}{2} x^{2a} y^2\right).
$$

**Theorem:** The number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid the pattern 111 is

$$
|\mathcal{S}_{m_1 m_2 \ldots m_k}(111)| = \begin{cases} 
\frac{(m_1+\cdots+m_k)!}{m_1!\cdots m_k!} & \text{if } m_i \leq 2 \quad \forall i \\
0 & \text{otherwise.}
\end{cases}
$$
**Proof:** Let $\sigma \in C_n^A(m; 111)$ and $A' = \{a_2, \ldots, a_d\}$. $\sigma$ avoids 111 $\Rightarrow a_1$ occurs 0, 1 or 2 times. Hence, for all $n, m \geq 0$,

$$C_n^A(m; 111) = C_n^{A'}(m; 111) + m C_{n-a_1; m-1}(111)$$

$$+ \binom{m}{2} C_{n-2a_1; m-2}(111)$$

Multiplying by $\frac{1}{m!} x^n y^m$ and summing over all $n, m \geq 0$ we get that

$$\sum_{m \geq 0} C_{111}^A(x; m) \frac{y^m}{m!} = \left(1 + x^{a_1}y + \frac{1}{2}x^{2a_1}y^2\right) \sum_{m \geq 0} C_{111}^{A'}(x; m) \frac{y^m}{m!}.$$ 

Since $\sum_{m \geq 0} C_{111}^{\{a_1\}}(x; m) \frac{y^m}{m!} = 1 + x^{a_1}y + \frac{1}{2}x^{2a_1}y^2$, we get the desired result by induction on $d$.

The result for multisets follows since the letter $i$ occurs either once or twice, and we can arrange the letters in $\frac{(m_1+\ldots+m_k)!}{m_1!\ldots m_k!}$ ways. $\blacksquare$
The patterns 121 and 112

Structure of compositions avoiding 121

- If more than one $a_1$ occurs in $\sigma$, they have to be consecutive
- Removing all $a_1$s from $\sigma$ gives $\sigma' \in \mathcal{A}'$ which avoids 121
- If more than one $a_2$ occurs in $\sigma'$, they have to be consecutive
- Deletion of parts $a_1$ through $a_j$ leaves a (possibly empty) composition $\sigma^{(j)}$ on parts $a_{j+1}$ through $a_d$ in which all parts $a_{j+1}$, if any, occur consecutively.

Structure of compositions avoiding 112

- Only leftmost $a_1$ can occur before any larger part in $\sigma$
- All other $a_1$s (excess $a_1$s) have to occur at the right end
- Deletion of parts $a_1$ through $a_j$ leaves a (possibly empty) composition $\sigma^{(j)}$ on parts $a_{j+1}$ through $a_d$ in which all excess $a_{j+1}$s occur at the end.
Bijection: \( \rho : C^A_{n;m}(121) \to C^A_{n;m}(112) \) \((*)\)

Let \( \sigma^{(0)} = \sigma \in C^A_{n;m}(121) \) and apply the following transformation of \( d \) steps.

- \( \sigma^{(j-1)} \) composition after \( j - 1 \) steps

- \( \sigma^{(j)} \) composition that results by cutting out the block of excess \( a_j \)'s and inserting it immediately before the final block of all smaller excess parts in \( \sigma^{(j-1)} \), or at the end of \( \sigma^{(j-1)} \) if there are no smaller excess parts.

Example:

\[
43221113 \to 43221311 \to 43213211 \to 43213211
\]
Theorem: Let \( A = \{a_1, \ldots, a_d\} \subset \mathbb{N} \). Then \( 112 \equiv^{\text{Wilf}} 121 \) and
\[
(1 - x^{a_1} y) C^A_{112}(x, y) = C^A'_{112}(x, y) + x^{a_1} y^2 \frac{\partial}{\partial y} C^A'_{112}(x, y),
\]
or, for all \( m \geq 1 \),
\[
C^A_{112}(x; m) = C^A'_{112}(x; m) + x^{a_1} C^A_{112}(x; m - 1) + (m - 1)x^{a_1} C^A'_{112}(x; m - 1),
\]
where \( C^A_{112}(x; 0) = 1 \) for any ordered set \( A \).

Theorem: The number of permutations of the multiset \( S \) that avoid the patterns 112 and 121, respectively, is
\[
|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112)| = |\mathfrak{S}_{m_1 m_2 \ldots m_k}(121)| = \prod_{j=2}^{k} (m_j + \cdots + m_k + 1).
\]
**Proof:** Wilf equivalence follows from the bijection \( \rho \). Let \( H^A_{112}(x, y) \) be the gf for compositions in \( C^A_{n;m}(112) \) which contain at least one part \( a_1 \). Write gf in two different ways:

\[
H^A_{112}(x, y) = C^A_{112}(x, y) - C^A'_{112}(x, y)
\]

\[
H^A_{112}(x, y) = x^{a_1} y C^A_{112}(x, y)
\]

\[
+ \sum_{n \geq a_1, m \geq 1} (m - 1)|C^A'_{n-a_1;m-1}(112)|x^n y^m
\]

Express the sum as a derivative and combine.

For multiset result, count 112 avoiding permutations: all excess 1’s occur at the end, so exactly one 1 occurs in \( \sigma' = \sigma_1 \ldots \sigma_{m+1-m_1} \) and \( \sigma \) avoids 112 \( \iff \sigma' \) avoids 112. The single 1 can occur in \((m_2 + \cdots + m_k + 1)\) positions. Thus

\[
|\mathcal{S}_{m_1 m_2 \ldots m_k}(112)| = (m_2 + \cdots + m_k + 1)|\mathcal{S}_{m_2 \ldots m_k}(112)|
\]

and formula follows using induction. \( \blacksquare \)
Example: Compositions avoiding 112 for $A = \{1, 2\}$

- $\{1, 1, 2, 3, 4, 6, 7, 10, 11, 15, 16, 21, 22, 28, 29, 36, 37, 45, 46, 55, 56\}$ for $n = 0 \ldots 20$.

- Occurs as sequence A055802; odd terms are triangle numbers (A000217), which also count # of permutations of $n$ which avoid 132 and have exactly one descent.

- $a(n) = |C_n^{\{1,2\}}(112)|$; $a(2i) = a(2i - 1) + 1$, $a(2i + 1) = a(2i) + i$.

- Compositions either end in 1 or if they end in 2, they can have at most one 1. Create recursively:
  1. Append a 1 to each composition of $n - 1$.
  2. If $n = 2i \rightarrow$ all 2’s (of which there is one)
     - If $n = 2i + 1 \rightarrow$ all 2’s and single 1 (of which there are $i$)

- $a(2i) = (i^2 + i + 2)/2$, $a(2i + 1) = (i + 1)(i + 2)/2$

- $a(n) = \frac{1}{16}(2n^2 + 6n + 11 - (2n - 5)(-1)^n)$
Example: Compositions avoiding 112 for $A = \{1, s\}$

- For $s = 4$: $\{1, 1, 1, 1, 2, 3, 3, 3, 4, 6, 6, 6, 7, 10, 10, 10, 11, 15, 15, 15, 16\}$ for $n = 0 \ldots 20$.

- now certain values repeat - extend from case $s = 2$

- $b(n) = |C_n^{1,s}(112)|$

- Compositions either end in 1 or if they end in $s$, they can have at most one 1. Create recursively:
  1. Append a 1 to each composition of $n - 1$.
  2. If $n = s \cdot i \rightarrow$ all $s$’s (of which there is one)
     - If $n = s \cdot i + 1 \rightarrow$ all $s$’s and single 1 (of which there are $i$)
  - If $n = s \cdot i + j$, $2 \leq j \leq (s - 1) \rightarrow$ no additional compositions ending in $s$

- $b(s \cdot i) = a(2i) = (i^2 + i + 2)/2$
  - $b(s \cdot i + j) = a(2i + 1) = (i + 1)(i + 2)/2$, $1 \leq j \leq (s - 1)$
The patterns 221 and 212

**Theorem:** Let $A = \{a_1, \ldots, a_d\}$ and $\tilde{A} = \{a_1, \ldots, a_{d-1}\}$ be ordered sets. Then patterns 221 and 212 are Wilf-equivalent on compositions, and

$$(1 - x^{a_d}y)C^{A}_{221}(x, y) = C^{\tilde{A}}_{221}(x, y) + x^{a_d}y^2 \frac{\partial}{\partial y} C^{\tilde{A}}_{221}(x, y).$$

**Theorem:** The number of permutations of the multiset $S = 1^{m_1}2^{m_2} \cdots k^{m_k}$ which avoid the patterns 221 and 212, respectively, is

$$|\mathcal{S}_{m_1m_2\ldots m_k}(221)| = |\mathcal{S}_{m_1m_2\ldots m_k}(212)|$$

$$= \prod_{j=1}^{k-1} (m_1 + \cdots + m_j + 1).$$
**Proof:** Either redo proofs for 112 and 121 with 112 replaced by 221, $a_1$ replaced by $a_d$, etc., or use a bijection based on the complement map

$$c(a_{i_1}a_{i_2}\ldots a_{i_m}) = a_{d+1-i_1}a_{d+1-i_2}\ldots a_{d+1-i_m}.$$ 

If $\sigma$ avoids 221, then $c(\sigma)$ avoids 112. Since $c$ is one-to-one, $c^{-1} \circ \rho^{-1} \circ c$ is a bijection between the sets $C_{n;m}^A(221)$ and $C_{n;m}^A(212)$.

\[
\begin{array}{ccc}
C_{n;m}^A(221) & \xrightarrow{c^{-1} \circ \rho^{-1} \circ c} & C_{n;m}^A(212) \\
C_{n';m}^A(112) & \xrightarrow{\rho^{-1}} & C_{n';m}^A(121) \\
\downarrow c & & \uparrow c^{-1}
\end{array}
\]
Example: Compositions avoiding 221 for $A = \{1, s\}$

- For $s = 2$: $\{1, 1, 2, 3, 5, 7, 10, 13, 17, 21, 26, 31, 37, 43, 50, 57, 65, 73, 82, 91, 101\}$ for $n = 0 \ldots 20$

- occurs as sequence A033638;
  $|C_n^{\{1,2\}}(221)| = \frac{1}{8}(2n^2 + 7 + (-1)^n)$.

- counts the number of $(3412,123)$-avoiding involutions in $\mathfrak{S}_n$
  (see Egge, 2004)

- for $s = 4$: $\{1, 1, 1, 1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 19, 22, 25, 29, 33, 37, 41, 46\}$ for $n = 0 \ldots 20$

- Recursive creation for $n = i \cdot s + \ell$
  1. Prepend a 1 to each composition of $n - 1$
  2. Start with $s$, place $n - j \cdot s$ 1’s, then the remaining $j - 1$ $s$’s.
     (of which there $i$ ones).

- $a(i \cdot s + \ell) = \frac{2+i(i-1)s+2i(\ell+1)}{2}$ for $n \geq 0$
**Pairs of Patterns**

- \( \sigma \) avoids \( \{\tau_1, \tau_2\} \Rightarrow \sigma \) avoids \( \{r(\tau_1), r(\tau_2)\} \).

- 21 possible pairs reduced to 13 symmetry classes of pairs

- \( \{111, 112\}, \{111, 121\}, \{111, 212\}, \{111, 221\} \)

- \( \{112, 121\}, \{122, 212\} \)

- \( \{112, 211\}, \{122, 221\} \)

- \( \{112, 212\}, \{122, 121\} \)

- \( \{121, 212\} \)

- \( \{112, 221\} \)

- \( \{112, 122\} \) - no result
The patterns \{111, 112\}, \{111, 121\} and \{111, 212\}, \{111, 221\}

Theorem: Let \(A = \{a_1, a_2, \ldots, a_d\} \subseteq \mathbb{N}\). \{111, 112\} \text{ Wilf} \equiv \{111, 121\} and \{111, 212\} \text{ Wilf} \equiv \{111, 221\}, and for all \(m \geq 0\),

\[
C_{111,112}^A(x; m) = C_{111,112}^{A'}(x; m) + mx^{a_1}C_{111,112}^{A'}(x; m - 1) \\
+ (m - 1)x^{2a_1}C_{111,112}^{A'}(x; m - 2),
\]

\[
C_{111,212}^A(x; m) = C_{111,212}^{\tilde{A}}(x; m) + mx^{a_d}C_{111,212}^{\tilde{A}}(x; m - 1) \\
+ (m - 1)x^{2a_d}C_{111,212}^{\tilde{A}}(x; m - 2).
\]

Proof: Avoiding 111 implies no, one or two \(a_i\)s. \(\rho\) preserves number of occurrences, so \(\rho|_{111} : C_{n;m}^A(111, 121) \to C_{n;m}^A(111, 112)\) is a bijection. One \(a_1\) can occur in any of \(m\) positions. If there are two copies of \(a_1\), second one is at the end because \(\sigma\) avoids 112, and the other \(a_1\) can occur at any of \(m - 1\) positions.
Theorem: The number of permutations of the multiset $S$ which avoid the patterns \{111, 112\} \text{ Wilf} \equiv \{111, 121\}$ and \{111, 212\} \text{ Wilf} \equiv \{111, 221\} are

$$|\mathcal{S}_{m_1m_2...m_k}(111, 112)| = \begin{cases} 0 & \text{if } \exists i \text{ with } m_i \geq 3 \\ \prod_{j=2}^{k}(m_j + \cdots + m_k + 1) & \text{otherwise.} \end{cases}$$

and

$$|\mathcal{S}_{m_1m_2...m_k}(111, 212)| = \begin{cases} 0 & \text{if } \exists i \text{ with } m_i \geq 3 \\ \prod_{j=1}^{k-1}(m_1 + \cdots + m_j + 1) & \text{otherwise.} \end{cases}$$

Proof: Wilf-equivalence follows from $\rho|_{111}$. Avoiding 111 restricts the multisets to those where $m_i \leq 2$ for all $i$. On these multisets, avoiding 112 (221, respectively) is the only restriction, and results for 112 and 221 give the stated results.
The patterns \{112, 121\} and \{122, 212\}

**Theorem:** Let \(A = \{a_1, a_2, \ldots, a_d\}\) be any ordered finite set of positive integers. Then for any \(m \geq 0\),

\[
C_{112,121}^A(x; m) = C_{112,121}^{A'}(x; m) + m x^{a_1} C_{112,121}^{A'}(x; m - 1) \\
+ \sum_{j=2}^{m} x^{ja_1} C_{112,121}^{A'}(x; m - j)
\]

and

\[
C_{122,212}^A(x; m) = C_{122,212}^{\tilde{A}}(x; m) + m x^{a_d} C_{122,212}^{\tilde{A}}(x; m - 1) \\
+ \sum_{j=2}^{m} x^{ja_d} C_{122,212}^{\tilde{A}}(x; m - j).
\]

**Proof:** Avoiding 112 and 121 simultaneously implies that for \(j > 1\), all copies of \(a_1\) have to appear in a block at the end. A single \(a_1\) can occur anywhere.
Theorem: The number of permutation of the multiset $S$ which avoid \{112, 121\} and \{122, 212\}, respectively, are

$$|\mathcal{S}_{m_1m_2\ldots m_k}(112, 121)| = \prod_{j=1}^{k-1} b_j, \text{ with } b_j = \begin{cases} (m_j + \cdots + m_k) & \text{if } m_j = 1 \\ 1 & \text{otherwise} \end{cases}$$

and

$$|\mathcal{S}_{m_1m_2\ldots m_k}(122, 212)| = \prod_{j=2}^{k} c_j, \text{ with } c_j = \begin{cases} (m_1 + \cdots + m_j) & \text{if } m_j = 1 \\ 1 & \text{otherwise} \end{cases}$$

Proof: Avoiding 112 and 121 simultaneously implies that for $j > 1$, all 1s have to appear in a block at the end. A single $a_1$ can occur in any of $m_2 + \ldots + m_k + 1$ positions.
The patterns $\{112, 211\}$ and $\{122, 221\}$

**Theorem:** Let $A = \{a_1, a_2, \ldots, a_d\}$ be any ordered finite set of positive integers. Then for all $m \geq 0$,

$$C_{112,211}^A(x; m) = C_{112,211}^A(x; m) + mx^{a_1}C_{112,211}^A(x; m - 1)$$

$$+ x^{2a_1}C_{112,211}^A(x; m - 2) + x^{m \cdot a_1}$$

and

$$C_{122,221}^A(x; m) = C_{122,221}^\tilde{A}(x; m) + mx^{a_d}C_{122,221}^\tilde{A}(x; m - 1)$$

$$+ x^{2a_d}C_{122,221}^\tilde{A}(x; m - 2) + x^{m \cdot a_d}.$$ 

**Proof:** $\sigma$ avoids $112$ and $211 \Rightarrow a_1$ occurs either 0, 1, 2, or $m$ times. If $j = 2$, $a_1$ occurs at the beginning and end.
Theorem: The number of permutation of the multiset $S$ which avoid \{112, 211\} and \{122, 221\}, respectively, are

$$|\mathcal{S}_{m_1 m_2 \ldots m_k}(112, 211)| = \prod_{j=1}^{k-1} b_j,$$

where

$$b_j = \begin{cases} 
(m_j + \cdots + m_k) & \text{if } m_j = 1 \\
1 & \text{if } m_j = 2 \\
0 & \text{if } m_j \geq 2
\end{cases}$$

and

$$|\mathcal{S}_{m_1 m_2 \ldots m_k}(122, 221)| = \prod_{j=2}^{k} c_j,$$

where

$$c_j = \begin{cases} 
(m_1 + \cdots + m_j) & \text{if } m_j = 1 \\
1 & \text{if } m_j = 2 \\
0 & \text{if } m_j \geq 2
\end{cases}$$
The patterns \{112, 212\} and \{122, 121\}

**Theorem:** Let \( A = \{a_1, a_2, \ldots, a_d\} \). Then

\[
C_A^{112,212}(x, y) = \prod_{j=1}^{d} \frac{1 + x^{a_j} y}{1 - x^{a_j} y} - \sum_{i=1}^{d} \left[ \frac{x^{a_i} y}{1 - x^{a_i} y} \prod_{j=1}^{i-1} \frac{1 + x^{a_j} y}{1 - x^{a_j} y} \right] 
\left( \frac{1}{1 - \sum_{U \sqcup V = A_i'} (C_U^{112,212}(x, y) - 1)(C_V^{112,212}(x, y) - 1)} \right)
\]

and

\[
C_A^{122,121}(x, y) = \prod_{j=1}^{d} \frac{1 + x^{a_j} y}{1 - x^{a_j} y} - \sum_{i=1}^{d} \left[ \frac{x^{a_i} y}{1 - x^{a_i} y} \prod_{j=i+1}^{d} \frac{1 + x^{a_j} y}{1 - x^{a_j} y} \right] 
\left( \frac{1}{1 - \sum_{U \sqcup V = \tilde{A}_i} (C_U^{122,121}(x, y) - 1)(C_V^{122,121}(x, y) - 1)} \right),
\]

where \( C_T^\emptyset(x, y) = 1 \), and \( U \sqcup V = D \) denotes the collection of sets \( U \) and \( V \) such that \( U \cup V = D \) and \( U \cap V = \emptyset \).
Proof: Focus on where $a_1$ occurs. Assume there are $j$ occurrences of $a_1$.

- $j = 0 \rightarrow C_{112,212}'^A(x, y)$

- $j \geq 1 : \sigma$ avoids $112 \Rightarrow$ excess $a_1$s at right end and
  \[ \sigma = \sigma^1 a_1 \sigma^2 a_1 \cdots a_1 \], where $\sigma^1, \sigma^2$, and block of $a_1$’s may be empty

- $\sigma$ avoids $212 \Rightarrow$ parts in $\sigma^1$ distinct from parts in $\sigma^2$. Thus, $\sigma \in C_{112,212}^A \iff \sigma^1 \in C_{112,212}^U$ and $\sigma^2 \in C_{112,212}^V$, with $U \sqcup V = A'$

- 3 cases to consider
  - $\sigma^1 = \sigma^2 = \emptyset$
  - exactly one of $\sigma^1$ and $\sigma^2$ is the empty set
  - neither $\sigma^1$ nor $\sigma^2$ is empty

Then use induction on $n$.  \[\blacksquare\]
Example: Compositions avoiding \{112, 212\} for $A = \{1, s\}$

- \(C_{112,212}^A(x) = \frac{1+x}{1-x} \cdot \frac{1+x^s}{1-x^s} - \frac{x}{1-x} \cdot \frac{1+x^s}{1-x^s} - \frac{x^s}{1-x^s} = \frac{1+x^{s+1}}{(1-x)(1-x^s)}\).

- \(s = 2\): \{1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\} for \(n = 0, \ldots, 20\)

- \(s = 4\): \{1, 1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 7, 7, 8, 9, 9, 9, 9, 10\}

- Recursive creation:
  1. Append a 1 to each composition of \(n - 1\) (ending in 1)
  2. If \(n = k \cdot s \rightarrow ss \ldots ss\)
     If \(n = k \cdot s + 1 \rightarrow 1ss \ldots ss\) (ending in s, distinct parts)

- \(|C_{112,212}^{\{1,s\}}(n)| = \begin{cases} 
  1 & \text{for } n = 0, \ldots, s - 1; \\
  2k & \text{for } n = k \cdot s, k \geq 1; \\
  2k + 1 & \text{for } n = k \cdot s + j, k \geq 1, 1 \leq j < s. 
\end{cases}\)
Theorem: The number of permutation of the multiset $S$ which avoid $\{112, 212\}$ and $\{122, 121\}$, respectively, are

$$|\mathcal{G}_{m_1 m_2 \ldots m_k} (112, 212)| = \sum_{Q \sqcup P = \{2, \ldots, k\}} |\mathcal{G}_{m_{q_1} \ldots m_{q_s}} (112, 212)||\mathcal{G}_{m_{p_1} \ldots m_{p_{k-s-1}}} (112, 212)|$$

and

$$|\mathcal{G}_{m_1 m_2 \ldots m_k} (122, 121)| = \sum_{Q \sqcup P = \{1, \ldots, k-1\}} |\mathcal{G}_{m_{q_1} \ldots m_{q_s}} (122, 121)||\mathcal{G}_{m_{p_1} \ldots m_{p_{k-s-1}}} (122, 121)|$$

where $Q = \{q_1, \ldots, q_s\}$, $P = \{p_1, \ldots, p_{k-1-s}\}$, and the number of permutations of the empty multiset are defined to be 1.
**The pattern** \{121, 212\}

**Theorem:** Let \( A \) be any ordered set of positive integers. Then

\[
C^A_{121, 212}(x, y) = 1 + \sum_{\emptyset \neq B \subseteq A} \left( |B|! \prod_{b \in B} \frac{x^b y}{1 - x^b y} \right),
\]

and the number of permutations of the multiset \( S \) which avoid \( \{121, 212\} \) is given by

\[
|\mathcal{S}_{m_1 m_2 \ldots m_k}(121, 212)| = k!.
\]

**Proof:** Avoiding 121 and 212 implies that all copies of each part \( a_{i_1} \) through \( a_{i_j} \) must be consecutive, i.e., \( \sigma \) is a concatenation of \( j \) constant strings.
The pattern \{122, 221\}

Theorem: Let \(A = \{a_1, a_2, \ldots, a_d\}\) be any ordered finite set of positive integers. Then

\[
C^A_{122, 221}(x, y) = 1 + \sum_{\emptyset \neq B \subset A} \left( |B|! \prod_{b \in B} (x^b y) \sum_{b' \in B} \frac{1}{1 - x^{b'} y} \right),
\]

and the number of permutations of the multiset \(S\) which avoid \{112, 221\} is given by

\[
|\mathcal{S}_{m_1 m_2 \ldots m_k}(112, 221)| = \begin{cases} 
0, & \text{if there exist } i, j \text{ such that } m_i, m_j \geq 2; \\
 k! & \text{otherwise.}
\end{cases}
\]

Proof: Let \(\sigma \in C^A_{n; m}(112, 221)\) and let \(j \leq m\) be the largest index such that \(\sigma_1, \sigma_2, \ldots, \sigma_j\) are all distinct. If \(j < m\), then \(\sigma_{j+1}\) repeats one of the preceding parts, and the parts to the right of \(\sigma_{j+1}\), if any, have to be equal to \(\sigma_{j+1}\) since \(\sigma\) avoids 112 and 221.
Patterns of arbitrary length

The pattern $\{11 \cdots 11\} = \langle 1 \rangle_\ell$

**Theorem:** For any $\ell \geq 1$ and any finite ordered set of positive integers $A$,

$$
\sum_{m \geq 0} C^A_{\langle 1 \rangle_\ell}(x; m) \frac{y^m}{m!} = \prod_{a \in A} \left( \sum_{j=0}^{\ell-1} \frac{x^j a^j y^j}{j!} \right),
$$

and the number of permutations of the multiset $S$ which avoid $\langle 1 \rangle_\ell$ is given by

$$
|\mathcal{S}_{m_1 m_2 \cdots m_k}(\langle 1 \rangle_\ell)| = \begin{cases} 
\frac{(m_1 + \cdots + m_k)!}{m_1! \cdots m_k!} & \text{if } m_i \leq \ell - 1 \quad \forall i; \\
0 & \text{otherwise.}
\end{cases}
$$
The pattern \( \{11 \cdots 121 \cdots 11\} \)

**Theorem:** Let \( A = \{a_1, \ldots, a_d\} \) be any finite ordered set of positive integers. Then \( v_{s,t} \overset{\text{Wilf}}{=} v_{s+t,0} \), and for all \( m \geq 1 \),

\[
C^A_{v_{s,t}}(x; m + 1) - x^{a_1} C^A_{v_{s,t}}(x; m) = \sum_{j=0}^{s+t-1} x^{ja_1} \binom{m}{j} C^A_{v_{s,t}}(x; m + 1 - j).
\]

The of permutations of the multiset \( S \) which avoid \( v_{s,t} \) is given by

\[
|\mathcal{S}_{m_1 m_2 \cdots m_k}(v_{s,t})| = \prod_{j=1}^{k-1} \left( m_{j+1} + \ldots + m_k + \min\{m_j, s + t - 1\} \right) \min\{m_j, s + t - 1\},
\]

where \( v_{s,t} \) has \( s \) 1’s on left and \( t \) 1’s on right of the single 2.
**Proof:** Let $j = \text{number of occurrences of } a_1$. Need to consider two cases:

- $j < s + t$ → $a_1$ is not part of $v_{s,t}$, i.e. they can occur in any of the $m$ positions

- $j \geq s + t$, then there are three possibilities:
  - $s, t \neq 0$ → leftmost $s - 1$ $a_1$’s and rightmost $t - 1$ $a_1$’s can occur anywhere, and the remaining $j - t - s + 2$ $a_1$’s have to occur as a block
  - $s = 0$ → block of $j - t + 1$ $a_1$’s have to occur as a block on the left
  - $t = 0$ → block of $j - s + 1$ $a_1$’s have to occur as a block on the right

In either case, can choose $s + t - 1$ positions out of the $m - (s + t - 1) + 1$
Thus,

\[
C^A_{v_s,t}(x; m) = \sum_{j=0}^{s+t-1} x^{ja_1} \binom{m}{j} C'_{v_s,t}(x; m-j)
\]

\[+\sum_{j=s+t}^{m} x^{ja_1} \binom{m-j+s+t-1}{s+t-1} C'_{v_s,t}(x; m-j)\]

which gives result after simplifying.

\[\blacklozenge\]

**Remark:**

1. Note that this theorem gives Wilf equivalence of 112, 121, and 211. 112 and 211 are in same symmetry class, but Wilf equivalence of 112 and 121 had to be proved.

2. There is also a bijection than can show Wilf equivalence directly, based on a generalization of the bijection \(\rho\).
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