Compositions and Multisets Restricted by Patterns of Length 3

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Pattern Avoidance

- Knuth - **Permutations** avoiding a **permutation pattern** of length 3 $\rightarrow$ Catalan numbers
- Simion & Schmidt - Permutations avoiding any given **set of permutation patterns** of length 3
- Burstein - **Words** avoiding a set of permutation patterns of length 3
- Burstein & Mansour - Words avoiding **patterns with repeated letters**
- Heubach & Mansour - **Compositions** avoiding **patterns of length 2**
- Savage & Wilf - Compositions avoiding **permutation patterns of length 3**

Things to come ...

- Compositions and multisets avoiding a **single pattern** of length 3 with repeated letters
- Compositions and multisets avoiding **pairs of patterns** of length 3 with repeated letters
- Compositions and multisets avoiding special patterns of arbitrary length: 111...111 & 11...121...11

Notation and Definitions

- $\mathbb{N} =$ set of all positive integers
- $A = \{a_1, a_2, \ldots, a_d\}$ ordered subset of $\mathbb{N}$; $A' = A - \{a_1\}$, $\tilde{A} = A - \{a_d\}$
- $[k] = \{1, 2, \ldots, k\}$; $[k]^n =$ set of all words of length n over $[k]$
- pattern τ = word in $[\ell]^k$ that contains each letter from $[\ell]$, possible with repetitions. Sets of patterns denoted by T
- $S = 1^{m_1} 2^{m_2} \ldots k^{m_k}, m_i > 0$ multiset
- $\mathfrak{S}_{m_1 m_2 \ldots m_k} =$ set of permutations on a multiset S
- $\sigma = \sigma_1 \sigma_2 \ldots \sigma_m =$ composition of $n \in \mathbb{N}$ with m parts where $\sum_{i=1}^{m} \sigma_i = n$
- C_n^A ($C_{n;m}^A$) = the set of all compositions of n with parts in A (m parts in A)

Notation and Definitions

- $\sigma \in C_n^A$ ($C_{n;m}^A$) contains τ if σ contains a subsequence isomorphic to τ. Otherwise, σ avoids τ and we write $\sigma \in C_n^A(\tau)$ ($\sigma \in C_{n;m}^A(\tau)$)
- T_1 and T_2 are Wilf-equivalent, denoted by $T_1 \overset{\text{Wilf}}{\equiv} T_2$, if $|C_{n;m}^A(T_1)| = |C_{n;m}^A(T_2)|$ for all A, m and n.
- reversal map $r(\tau) = r(\tau_1\tau_2\ldots\tau_k) = \tau_k\tau_{k-1}\ldots\tau_1$; $r(\tau) \overset{\text{Wilf}}{\equiv} \tau$
- $\{\tau, r(\tau)\}$ = symmetry class of τ
- Generating functions
  - $C_T^A(x;m) = \sum_{n\geq 0} |C_{n;m}^A(T)|x^n$
  - $C_T^A(x,y) = \sum_{m\geq 0} C_T^A(x;m)y^m$
  - $C_T^A(x) = C_T^A(x,1) = \sum_{n\geq 0} |C_n^A(T)|x^n$.

Single Patterns of Length 3

- **111**
- **121** and **112**
- **221** and **212**
- 123, 132 and 213 (permutation patterns)
  Wilf equivalence for compositions and multisets and generating function for compositions (Savage and Wilf)

The pattern 111

Theorem: Let $A = \{a_1, \ldots, a_d\}$ be any ordered finite or infinite set of positive integers. Then

$$\sum_{m \geq 0} C_{111}^{A}(x;m) \frac{y^m}{m!} = \prod_{a \in A} \left(1 + x^a y + \frac{1}{2} x^{2a} y^2\right).$$

Theorem: The number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid the pattern 111 is

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(111)| = \begin{cases} \frac{(m_1+\cdots+m_k)!}{m_1!\cdots m_k!} & \text{if } m_i \leq 2 \quad \forall i \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Let $\sigma \in C^A_{n;m}(111)$ and $A' = \{a_2, \ldots, a_d\}$. σ avoids 111 $\Rightarrow$ a_1 occurs 0, 1 or 2 times. Hence, for all $n, m \geq 0$,

$$C^A_{n;m}(111) = C^{A'}_{n;m}(111) + m\, C^{A'}_{n-a_1;m-1}(111) + \binom{m}{2} C^{A'}_{n-2a_1;m-2}(111)$$

Multiplying by $\frac{1}{m!}x^n y^m$ and summing over all $n, m \geq 0$ we get that

$$\sum_{m\geq 0} C^A_{111}(x;m)\frac{y^m}{m!} = \left(1 + x^{a_1}y + \frac{1}{2}x^{2a_1}y^2\right)\sum_{m\geq 0} C^{A'}_{111}(x;m)\frac{y^m}{m!}.$$

Since $\sum_{m\geq 0} C^{\{a_1\}}_{111}(x;m)\frac{y^m}{m!} = 1 + x^{a_1}y + \frac{1}{2}x^{2a_1}y^2$, we get the desired result by induction on d.

The result for multisets follows since the letter i occurs either once or twice, and we can arrange the letters in $\frac{(m_1+\cdots+m_k)!}{m_1!\cdots m_k!}$ ways. ∎

The patterns 121 and 112

Structure of compositions avoiding 121

- If more than one a_1 occurs in σ, they have to be consecutive
- Removing all a_1s from σ gives σ' $\in A'$ which avoids 121
- If more than one a_2 occurs in σ', they have to be consecutive
- Deletion of parts a_1 through a_j leaves a (possibly empty) composition $\sigma^{(j)}$ on parts a_{j+1} through a_d in which all parts a_{j+1}, if any, occur consecutively.

Structure of compositions avoiding 112

- Only leftmost a_1 can occur before any larger part in σ
- All other a_1s (excess a_1s) have to occur at the right end
- Deletion of parts a_1 through a_j leaves a (possibly empty) composition $\sigma^{(j)}$ on parts a_{j+1} through a_d in which all excess a_{j+1}s occur at the end.

Bijection: $\rho : C^A_{n;m}(121) \to C^A_{n;m}(112)$ $(*)$

Let $\sigma^{(0)} = \sigma \in C^A_{n;m}(121)$ and apply the following transformation of d steps.

- $\sigma^{(j-1)}$ composition after $j-1$ steps
- $\sigma^{(j)}$ composition that results by cutting out the block of excess a_j's and inserting it immediately before the final block of all smaller excess parts in $\sigma^{(j-1)}$, or at the end of $\sigma^{(j-1)}$ if there are no smaller excess parts.

Example:

$$43221113 \to 43221311 \to 43213211 \to 43213211$$

Theorem: Let $A = \{a_1, \ldots, a_d\} \subset \mathbb{N}$. Then $112 \stackrel{\text{Wilf}}{\equiv} 121$ and

$$(1 - x^{a_1} y) C_{112}^{A}(x, y) = C_{112}^{A'}(x, y) + x^{a_1} y^2 \frac{\partial}{\partial y} C_{112}^{A'}(x, y),$$

or, for all $m \geq 1$,

$$C_{112}^{A}(x; m) = C_{112}^{A'}(x; m) + x^{a_1} C_{112}^{A}(x; m-1) + (m-1) x^{a_1} C_{112}^{A'}(x; m-1),$$

where $C_{112}^{A}(x; 0) = 1$ for any ordered set A.

Theorem: The number of permutations of the multiset S that avoid the patterns 112 and 121, respectively, is

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112)| = |\mathfrak{S}_{m_1 m_2 \ldots m_k}(121)| = \prod_{j=2}^{k} (m_j + \cdots + m_k + 1).$$

Proof: Wilf equivalence follows from the bijection ρ. Let $H^A_{112}(x,y)$ be the gf for compositions in $C^A_{n;m}(112)$ which contain at least one part a_1. Write gf in two different ways:

$$H^A_{112}(x,y) = C^A_{112}(x,y) - C^{A'}_{112}(x,y)$$

$$H^A_{112}(x,y) = x^{a_1} y\, C^A_{112}(x,y) + \sum_{n \geq a_1,\, m \geq 1} (m-1)|C^{A'}_{n-a_1;m-1}(112)|x^n y^m$$

Express the sum as a derivative and combine.

For multiset result, count 112 avoiding permutations: all excess 1's occur at the end, so exactly one 1 occurs in $\sigma' = \sigma_1 \ldots \sigma_{m+1-m_1}$ and σ avoids 112 $\iff$ σ' avoids 112. The single 1 can occur in $(m_2 + \cdots + m_k + 1)$ positions. Thus

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112)| = (m_2 + \cdots + m_k + 1)|\mathfrak{S}_{m_2 \ldots m_k}(112)|$$

and formula follows using induction. ∎

Example: Compositions avoiding 112 for $A = \{1, 2\}$

- {1, 1, 2, 3, 4, 6, 7, 10, 11, 15, 16, 21, 22, 28, 29, 36, 37, 45, 46, 55, 56} for $n = 0 \ldots 20$.
- occurs as sequence A055802; odd terms are triangle numbers (A000217), which also count # of permutations of n which avoid 132 and have exactly one descent
- $a(n) = |C_n^{\{1,2\}}(112)|$; $a(2i) = a(2i-1)+1$, $a(2i+1) = a(2i)+i$.
- Compositions either end in 1 or if they end in 2, they can have at most one 1. Create recursively:
  1. Append a 1 to each composition of $n-1$.
  2. If $n = 2i \rightarrow$ all 2's (of which there is one)
     If $n = 2i+1 \rightarrow$ all 2's and single 1 (of which there are i)
- $a(2i) = (i^2+i+2)/2$, $a(2i+1) = (i+1)(i+2)/2$
- $a(n) = \frac{1}{16}(2n^2+6n+11-(2n-5)(-1)^n)$

Example: Compositions avoiding 112 for $A = \{1, s\}$

- For $s = 4$: $\{1, 1, 1, 1, 2, 3, 3, 3, 4, 6, 6, 6, 7, 10, 10, 10, 11, 15, 15, 15, 16\}$ for $n = 0 \ldots 20$.
- now certain values repeat - extend from case $s = 2$
- $b(n) = |C_n^{\{1,s\}}(112)|$
- Compositions either end in 1 or if they end in s, they can have at most one 1. Create recursively:
  1. Append a 1 to each composition of $n - 1$.
  2. If $n = s \cdot i \longrightarrow$ all s's (of which there is one)
     If $n = s \cdot i + 1 \longrightarrow$ all s's and single 1 (of which there are i)
- If $n = s \cdot i + j$, $2 \leq j \leq (s - 1) \longrightarrow$ no additional compositions ending in s
- $b(s \cdot i) = a(2i) = (i^2 + i + 2)/2$
  $b(s \cdot i + j) = a(2i + 1) = (i + 1)(i + 2)/2$, $1 \leq j \leq (s - 1)$

The patterns 221 and 212

Theorem: Let $A = \{a_1, \ldots, a_d\}$ and $\tilde{A} = \{a_1, \ldots, a_{d-1}\}$ be ordered sets. Then patterns 221 and 212 are Wilf-equivalent on compositions, and

$$(1 - x^{a_d} y) C_{221}^{A}(x, y) = C_{221}^{\tilde{A}}(x, y) + x^{a_d} y^2 \frac{\partial}{\partial y} C_{221}^{\tilde{A}}(x, y).$$

Theorem: The number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid the patterns 221 and 212, respectively, is

$$\begin{aligned} |\mathfrak{S}_{m_1 m_2 \ldots m_k}(221)| &= |\mathfrak{S}_{m_1 m_2 \ldots m_k}(212)| \\ &= \prod_{j=1}^{k-1} (m_1 + \cdots + m_j + 1). \end{aligned}$$

Proof: Either redo proofs for 112 and 121 with 112 replaced by 221, a_1 replaced by a_d, etc., or use a bijection based on the complement map

$$c(a_{i_1}a_{i_2}\dots a_{i_m}) = a_{d+1-i_1}a_{d+1-i_2}\dots a_{d+1-i_m}.$$

If σ avoids 221, then $c(\sigma)$ avoids 112. Since c is one-to-one, $c^{-1}\circ\rho^{-1}\circ c$ is a bijection between the sets $C^A_{n;m}(221)$ and $C^A_{n;m}(212)$.

$$\begin{array}{ccc} C^A_{n;m}(221) & \xrightarrow{c^{-1}\circ\rho^{-1}\circ c} & C^A_{n;m}(212) \\ c\downarrow & & \uparrow c^{-1} \\ C^A_{n';m}(112) & \xrightarrow[\rho^{-1}]{} & C^A_{n';m}(121) \end{array}$$

Example: Compositions avoiding 221 for $A = \{1, s\}$

- For $s = 2$: $\{1, 1, 2, 3, 5, 7, 10, 13, 17, 21, 26, 31, 37, 43, 50, 57, 65, 73, 82, 91, 101\}$ for $n = 0 \ldots 20$
- occurs as sequence A033638;
  $|C_n^{\{1,2\}}(221)| = \frac{1}{8}(2n^2 + 7 + (-1)^n)$.
- counts the number of (3412,123)-avoiding involutions in $\mathfrak{S}_n$ (see Egge, 2004)
- for $s = 4$: $\{1, 1, 1, 1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 19, 22, 25, 29, 33, 37, 41, 46\}$ for $n = 0 \ldots 20$
- Recursive creation for $n = i \cdot s + \ell$
  1. Prepend a 1 to each composition of $n - 1$
  2. Start with s, place $n - j \cdot s$ 1's, then the remaining $j - 1$ s's. (of which there i ones).
- $a(i \cdot s + \ell) = \frac{2+i(i-1)s+2i(\ell+1)}{2}$ for $n \geq 0$

Pairs of Patterns

- σ avoids $\{\tau_1, \tau_2\} \Rightarrow \sigma$ avoids $\{r(\tau_1), r(\tau_2)\}$.
- 21 possible pairs reduced to 13 symmetry classes of pairs
- $\{111, 112\}, \{111, 121\}$, $\{111, 212\}, \{111, 221\}$
- $\{112, 121\}, \{122, 212\}$
- $\{112, 211\}, \{122, 221\}$
- $\{112, 212\}$, $\{122, 121\}$
- $\{121, 212\}$
- $\{112, 221\}$
- $\{112, 122\}$ - no result

The patterns $\{111,112\},\{111,121\}$ and $\{111,212\}$, $\{111,221\}$

Theorem: Let $A=\{a_1,a_2,\ldots,a_d\}\subseteq\mathbb{N}$. $\{111,112\}\overset{\text{Wilf}}{\equiv}\{111,121\}$ and $\{111,212\}\overset{\text{Wilf}}{\equiv}\{111,221\}$, and for all $m\geq 0$,

$$\begin{aligned}C_{111,112}^{A}(x;m)&=C_{111,112}^{A'}(x;m)+m\,x^{a_1}C_{111,112}^{A'}(x;m-1)\\&\quad+(m-1)x^{2a_1}C_{111,112}^{A'}(x;m-2),\\C_{111,212}^{A}(x;m)&=C_{111,212}^{\tilde{A}}(x;m)+mx^{a_d}C_{111,212}^{\tilde{A}}(x;m-1)\\&\quad+(m-1)x^{2a_d}C_{111,212}^{\tilde{A}}(x;m-2).\end{aligned}$$

Proof: Avoiding 111 implies no, one or two a_is. ρ preserves number of occurrences, so $\rho_{|111}:C_{n;m}^{A}(111,121)\to C_{n;m}^{A}(111,112)$ is a bijection. One a_1 can occur in any of m positions. If there are two copies of a_1, second one is at the end because σ avoids 112, and the other a_1 can occur at any of $m-1$ positions.

Theorem: The number of permutations of the multiset S which avoid the patterns $\{111, 112\} \overset{\text{Wilf}}{\equiv} \{111, 121\}$ and $\{111, 212\} \overset{\text{Wilf}}{\equiv} \{111, 221\}$ are

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(111, 112)| = \begin{cases} 0 & \text{if } \exists i \text{ with } m_i \geq 3 \\ \prod_{j=2}^{k}(m_j + \cdots + m_k + 1) & \text{otherwise.} \end{cases}$$

and

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(111, 212)| = \begin{cases} 0 & \text{if } \exists i \text{ with } m_i \geq 3 \\ \prod_{j=1}^{k-1}(m_1 + \cdots + m_j + 1) & \text{otherwise.} \end{cases}$$

Proof: Wilf-equivalence follows from $\rho_{|_{111}}$. Avoiding 111 restricts the multisets to those where $m_i \leq 2$ for all i. On these multisets, avoiding 112 (221, respectively) is the only restriction, and results for 112 and 221 give the stated results.

The patterns $\{112, 121\}$ and $\{122, 212\}$

Theorem: Let $A = \{a_1, a_2, \ldots, a_d\}$ be any ordered finite set of positive integers. Then for any $m \geq 0$,

$$C^{A}_{112,121}(x; m) = C^{A'}_{112,121}(x; m) + m\, x^{a_1} C^{A'}_{112,121}(x; m-1) + \sum_{j=2}^{m} x^{j a_1} C^{A'}_{112,121}(x; m-j)$$

and

$$C^{A}_{122,212}(x; m) = C^{\tilde{A}}_{122,212}(x; m) + m\, x^{a_d} C^{\tilde{A}}_{122,212}(x; m-1) + \sum_{j=2}^{m} x^{j a_d} C^{\tilde{A}}_{122,212}(x; m-j).$$

Proof: Avoiding 112 and 121 simultaneously implies that for $j > 1$, all copies of a_1 have to appear in a block at the end. A single a_1 can occur anywhere. ■

Theorem: The number of permutation of the multiset S which avoid $\{112, 121\}$ and $\{122, 212\}$, respectively, are

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112, 121)| = \prod_{j=1}^{k-1} b_j, \text{ with } b_j = \begin{cases} (m_j + \cdots + m_k) & \text{if } m_j = 1 \\ 1 & \text{otherwise} \end{cases}$$

and

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(122, 212)| = \prod_{j=2}^{k} c_j, \text{ with } c_j = \begin{cases} (m_1 + \cdots + m_j) & \text{if } m_j = 1 \\ 1 & \text{otherwise} \end{cases}$$

Proof: Avoiding 112 and 121 simultaneously implies that for $j > 1$, all 1s have to appear in a block at the end. A single a_1 can occur in any of $m_2 + ... + m_k + 1$ positions. ■

The patterns $\{112, 211\}$ and $\{122, 221\}$

Theorem: Let $A = \{a_1, a_2, \ldots, a_d\}$ be any ordered finite set of positive integers. Then for all $m \geq 0$,

$$C^{A}_{112,211}(x;m) = C^{A'}_{112,211}(x;m) + m\,x^{a_1} C^{A'}_{112,211}(x;m-1) + x^{2a_1} C^{A'}_{112,211}(x;m-2) + x^{m\,a_1}$$

and

$$C^{A}_{122,221}(x;m) = C^{\tilde{A}}_{122,221}(x;m) + m\,x^{a_d} C^{\tilde{A}}_{122,221}(x;m-1) + x^{2a_d} C^{\tilde{A}}_{122,221}(x;m-2) + x^{m\,a_d}.$$

Proof: σ avoids 112 and 211 $\Rightarrow$ a_1 occurs either 0, 1, 2, or m times. If $j = 2$, a_1 occurs at the beginning and end.

Theorem: The number of permutation of the multiset S which avoid $\{112, 211\}$ and $\{122, 221\}$, respectively, are

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112, 211)| = \prod_{j=1}^{k-1} b_j, \quad \text{where}$$

$$b_j = \begin{cases} (m_j + \cdots + m_k) & \text{if } m_j = 1 \\ 1 & \text{if } m_j = 2 \\ 0 & \text{if } m_j \geq 2 \end{cases}$$

and

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(122, 221)| = \prod_{j=2}^{k} c_j, \quad \text{where}$$

$$c_j = \begin{cases} (m_1 + \cdots + m_j) & \text{if } m_j = 1 \\ 1 & \text{if } m_j = 2 \\ 0 & \text{if } m_j \geq 2 \end{cases}$$

The patterns $\{112, 212\}$ and $\{122, 121\}$

Theorem: Let $A = \{a_1, a_2, \ldots, a_d\}$. Then

$$C^{A}_{112,212}(x,y) = \prod_{j=1}^{d} \frac{1+x^{a_j}y}{1-x^{a_j}y} - \sum_{i=1}^{d} \left[\frac{x^{a_i}y}{1-x^{a_i}y} \prod_{j=1}^{i-1} \frac{1+x^{a_j}y}{1-x^{a_j}y} \cdot \left(1 - \sum_{U \sqcup V = A'_i} (C^{U}_{112,212}(x,y)-1)(C^{V}_{112,212}(x,y)-1)\right)\right]$$

and

$$C^{A}_{122,121}(x,y) = \prod_{j=1}^{d} \frac{1+x^{a_j}y}{1-x^{a_j}y} - \sum_{i=1}^{d} \left[\frac{x^{a_i}y}{1-x^{a_i}y} \prod_{j=i+1}^{d} \frac{1+x^{a_j}y}{1-x^{a_j}y} \cdot \left(1 - \sum_{U \sqcup V = \tilde{A}_i} (C^{U}_{122,121}(x,y)-1)(C^{V}_{122,121}(x,y)-1)\right)\right],$$

where $C^{\varnothing}_{T}(x,y) = 1$, and $U \sqcup V = D$ denotes the collection of sets U and V such that $U \cup V = D$ and $U \cap V = \emptyset$.

Proof: Focus on where a_1 occurs. Assume there are j occurrences of a_1.

- $j = 0 \rightarrow C^{A'}_{112,212}(x, y)$
- $j \geq 1$: σ avoids 112 $\Rightarrow$ excess a_1s at right end and $\sigma = \sigma^1 a_1 \sigma^2 a_1 \cdots a_1$, where σ^1, σ^2, and block of a_1's may be empty
- σ avoids 212 $\Rightarrow$ parts in σ^1 distinct from parts in σ^2. Thus, $\sigma \in C^A_{112,212} \iff \sigma^1 \in C^U_{112,212}$ and $\sigma^2 \in C^V_{112,212}$, with $U \sqcup V = A'$
- 3 cases to consider
  - $\sigma^1 = \sigma^2 = \emptyset$
  - exactly one of σ^1 and σ^2 is the empty set
  - neither σ^1 nor σ^2 is empty

Then use induction on n. ■

Example: Compositions avoiding $\{112, 212\}$ for $A = \{1, s\}$

- $C^{A}_{112,212}(x) = \frac{1+x}{1-x} \cdot \frac{1+x^s}{1-x^s} - \frac{x}{1-x} \cdot \frac{1+x^s}{1-x^s} - \frac{x^s}{1-x^s} = \frac{1+x^{s+1}}{(1-x)(1-x^s)}$.
- $s = 2$: {1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20} for $n = 0, \ldots, 20$
- $s = 4$: {1, 1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 7, 8, 9, 9, 9, 10}
- Recursive creation:
  1. Append a 1 to each composition of $n - 1$ (ending in 1)
  2. If $n = k \cdot s \to ss \ldots ss$

     If $n = k \cdot s + 1 \to 1ss \ldots ss$ (ending in s, distinct parts)
- $|C_n^{\{1,s\}}(112, 212)| = \begin{cases} 1 & \text{for } n = 0, \ldots, s-1; \\ 2k & \text{for } n = k \cdot s,\ k \geq 1; \\ 2k+1 & \text{for } n = k \cdot s + j,\ k \geq 1, 1 \leq j < s. \end{cases}$

Theorem: The number of permutation of the multiset S which avoid $\{112, 212\}$ and $\{122, 121\}$, respectively, are

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112, 212)| = \sum_{Q \sqcup P = \{2,\ldots,k\}} |\mathfrak{S}_{m_{q_1} \ldots m_{q_s}}(112, 212)||\mathfrak{S}_{m_{p_1} \ldots m_{p_{k-s-1}}}(112, 212)|$$

and

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(122, 121)| = \sum_{Q \sqcup P = \{1,\ldots,k-1\}} |\mathfrak{S}_{m_{q_1} \ldots m_{q_s}}(122, 121)||\mathfrak{S}_{m_{p_1} \ldots m_{p_{k-s-1}}}(122, 121)|$$

where $Q = \{q_1, \ldots, q_s\}$, $P = \{p_1, \ldots, p_{k-1-s}\}$, and the number of permutations of the empty multiset are defined to be 1.

The pattern $\{121, 212\}$

Theorem: Let A be any ordered set of positive integers. Then

$$C^{A}_{121,212}(x,y) = 1 + \sum_{\varnothing \neq B \subset A} \left(|B|! \prod_{b \in B} \frac{x^b y}{1 - x^b y} \right),$$

and the number of permutations of the multiset S which avoid $\{121, 212\}$ is given by

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(121, 212)| = k!.$$

Proof: Avoiding 121 and 212 implies that all copies of each part a_{i_1} through a_{i_j} must be consecutive, i.e., σ is a concatenation of j constant strings. ■

The pattern $\{122, 221\}$

Theorem: Let $A = \{a_1, a_2, \ldots, a_d\}$ be any ordered finite set of positive integers. Then

$$C^A_{122,221}(x, y) = 1 + \sum_{\varnothing \neq B \subset A} \left(|B|! \prod_{b \in B} (x^b y) \sum_{b' \in B} \frac{1}{1 - x^{b'} y} \right),$$

and the number of permutations of the multiset S which avoid $\{112, 221\}$ is given by

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112, 221)| = \begin{cases} 0, & \text{if there exist } i, j \text{ such that } m_i, m_j \geq 2; \\ k! & \text{otherwise.} \end{cases}$$

Proof: Let $\sigma \in C^A_{n;m}(112, 221)$ and let $j \leq m$ be the largest index such that $\sigma_1, \sigma_2, \ldots, \sigma_j$ are all distinct. If $j < m$, then σ_{j+1} repeats one of the preceding parts, and the parts to the right of σ_{j+1}, if any, have to be equal to σ_{j+1} since σ avoids 112 and 221. ■

Patterns of arbitrary length

The pattern $\{11\cdots11\} = \langle 1\rangle_\ell$

Theorem: For any $\ell \geq 1$ and any finite ordered set of positive integers A,

$$\sum_{m\geq 0} C^A_{\langle 1\rangle_\ell}(x;m)\frac{y^m}{m!} = \prod_{a\in A}\left(\sum_{j=0}^{\ell-1}\frac{x^{ja}y^j}{j!}\right),$$

and the number of permutations of the multiset S which avoid $\langle 1\rangle_\ell$ is given by

$$|\mathfrak{S}_{m_1m_2\ldots m_k}(\langle 1\rangle_\ell)| = \begin{cases} \frac{(m_1+\cdots+m_k)!}{m_1!\cdots m_k!} & \text{if } m_i \leq \ell-1 \quad \forall i; \\ 0 & \text{otherwise.}\end{cases}$$

The pattern $\{11\cdots121\cdots11\}$

Theorem:Let $A=\{a_1,\dots,a_d\}$ be any finite ordered set of positive integers. Then $v_{s,t} \overset{\text{Wilf}}{\equiv} v_{s+t,0}$, and for all $m\geq 1$,

$$C^A_{v_{s,t}}(x;m+1)-x^{a_1}C^A_{v_{s,t}}(x;m)=\sum_{j=0}^{s+t-1}x^{ja_1}\binom{m}{j}C^{A'}_{v_{s,t}}(x;m+1-j).$$

The of permutations of the multiset S which avoid $v_{s,t}$ is given by

$$|\mathfrak{S}_{m_1m_2\dots m_k}(v_{s,t})|=\prod_{j=1}^{k-1}\binom{m_{j+1}+\dots+m_k+\min\{m_j,s+t-1\}}{\min\{m_j,s+t-1\}},$$

where $v_{s,t}$ has s 1's on left and t 1's on right of the single 2.

Proof: Let j = number of occurrences of a_1. Need to consider two cases:

- $j < s + t \to a_1$ is not part of $v_{s,t}$, i.e. they can occur in any of the m positions
- $j \geq s + t$, then there are three possibilities:
  - $s, t \neq 0 \to$ leftmost $s - 1$ a_1's and rightmost $t - 1$ a_1's can occur anywhere, and the remaining $j - t - s + 2$ a_1's have to occur as a block
  - $s = 0 \to$ block of $j - t + 1$ a_1's have to occur as a block on the left
  - $t = 0 \to$ block of $j - s + 1$ a_1's have to occur as a block on the right

  In either case, can choose $s + t - 1$ positions out of the $m - (s + t - 1) + 1$

Thus,

$$C^{A}_{v_{s,t}}(x;m) = \sum_{j=0}^{\mathbf{s}+\mathbf{t}-1} x^{ja_1}\binom{m}{j}C^{A'}_{v_{s,t}}(x;m-j) + \sum_{j=\mathbf{s}+\mathbf{t}}^{m} x^{ja_1}\binom{m-j+\mathbf{s}+\mathbf{t}-1}{\mathbf{s}+\mathbf{t}-1}C^{A'}_{v_{s,t}}(x;m-j)$$

which gives result after simplifying. ■

Remark:

1. Note that this theorem gives Wilf equivalence of 112, 121, and 211. 112 and 211 are in same symmetry class, but Wilf equivalence of 112 and 121 had to be proved.

2. There is also a bijection than can show Wilf equivalence directly, based on a generalization of the bijection ρ.

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