THE GAME CREATION OPERATOR

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Outline

- Basic background on combinatorial games
- Definition of subtraction games
- Some examples: Nim, Wythoff
- History of the game creation operator
- Our results
- Future work

The Basics

- A two-player game is called a **combinatorial game** if there is no randomness involved and all possible moves are known to each player.
- A combinatorial game is called **impartial** if both players have the same allowed moves

**Examples:**

- Under **normal play**, the last player to move wins. Under **misère play**, the last player to move loses.
Main Question:

Who wins in a combinatorial game from a specific position, assuming both players play optimally?

Subtraction Games

- A subtraction or take-away game is played on one or more stacks of tokens
- Positions are described as vectors of stack heights
- The subtraction set $M$ consists of the possible moves in the form of subtraction vectors. A move can be used as long as it does not result in negative stack height(s)

\[
\begin{array}{c}
\text{Take one token from stack 1} \\
(5, 3, 2, 1) \quad \rightarrow \quad (1, 0, 0, 0)
\end{array}
\]
Subtraction Sets

Examples:
- **Nim** on one stack \( M = \{1, 2, 3, \ldots\} \)

- **Wythoff** is played on two stacks. Can either take
  - one or more tokens from one stack, or
  - the same number from both stacks.

\[ M = \{(1,0), (2,0), \ldots, (0,1), (0,2), \ldots, (1,1), (2,2), \ldots\} \]

Impartial Games

Only two possible outcome classes:
- **Losing** positions
- **Winning** positions

Characterization of positions
- From a losing position, all allowed moves lead to a winning position
- From a winning position, there is at least one move to a losing position.
- In misère play, the terminal positions are winning positions
Recursive Determination of Outcome Class

- Game $\text{M} = \{4, 7, 11\}$
- We will color winning and losing positions
- The terminal positions are 0, 1, 2, 3
- Pattern that emerges is an alternating sequence of 4 losing positions followed by 11 winning positions (after the terminal positions in the beginning)

★-Operator

Observation: For subtraction games, positions and allowed moves have the same structure! This allows us to iteratively create new games.

The ★-operator is defined as follows:
- We start with a subtraction game $\text{M}$ that is described by the allowed moves.
- We compute the set of losing positions, $\text{L(M)}$
- The losing positions of $\text{M}$ become the moves for the game $\text{M}^*$
- Notation: $\text{M}^0 = \text{M}$, $\text{M}^n = (\text{M}^{n-1})^*$
- $\text{M}$ is reflexive if $\text{M} = \text{M}^*$
How did the ★-Operator come about?

**WYTHOFF**

- The losing positions of WYTHOFF (under normal play) are closely related to the golden ratio \( \varphi = \frac{1+\sqrt{5}}{2} \):

\[
\mathcal{L} = \{([n \cdot \varphi], [n \cdot \varphi] + n)|n \geq 0}\}
\]

- We only list positions of the form \((x,y)\), but by symmetry, \((y,x)\) is also a losing position.
Visualization of the Losing Positions

Recursive Creation of the Losing Positions

- The losing positions can also be created recursively.

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_n</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>b_n</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>15</td>
<td>18</td>
</tr>
</tbody>
</table>

- Let $a_n =$ the smallest non-negative integer not yet used and set $b_n = a_n + n$. Repeat.
- By creation, sequences \{a_n\} and \{b_n\} are **complementary**, in fact, they are homogenous **Beatty sequences**.
Complementary Beatty Sequences

From American Mathematical Monthly, 33 (3): 159

**Duchêne-Rigo Conjecture**: Every complementary pair of homogeneous Beatty sequences forms the set of losing positions for some invariant impartial game.

This conjecture was proved by Larsson, Hegarty and Fraenkel using the game creation operator (★-operator)
Questions for Misère-Play ★-Operator

• **Question 1**: Does the misère-play ★-operator converge (point-wise)?

• **Question 2**: What feature(s) of $\mathbb{M}$ determines the limit game for its sequence?

• **Question 3**: Limit games are (by definition) reflexive. What is the structure of reflexive games and/or limit games (if they exist)?

• **Question 4**: How quickly does convergence occur?
Example for One Stack

★-operator applied five times to initial game

\[ M^0 = \{4, 7, 11\} \quad G^0 = \{4, 9\} \]

Observations from Example

- Looks like there is convergence (fixed point) for each of the games
- Limit games seem to have a periodic structure: blocks of moves alternate with blocks of non-moves
- \( M^0 = \{4, 7, 11\} \) and \( G^0 = \{4, 9\} \) seem to have the same limit game

**Question**: What do the two sets \( M^0 \) and \( G^0 \) have in common?

**Answer**: The minimal element, \( k = 4 \).
Q1: Convergence Result

**Theorem**
Starting from any game $M$ on $d$ stacks, the sequence of games created by the misère-play ★-operator converges to a (reflexive) limit game $M^\infty$.

**Proof idea:** (for $d$ stacks)
- Positions become fixed either as moves or non-moves from “smaller to larger”. There are four possibilities:

<table>
<thead>
<tr>
<th>Move in $M^i$</th>
<th>Move in $M^{i+1}$</th>
<th>Non-move in $M^{i+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed as move</td>
<td>Erased as move</td>
<td></td>
</tr>
<tr>
<td>Introduced as move</td>
<td>Fixed as non-move</td>
<td></td>
</tr>
</tbody>
</table>
- Show that smallest element not yet fixed becomes fixed.
Proof by Picture

- Not all positions switch from non-move to move:
Q2: Which Feature of $M$ Determines $M^\infty$?

**Theorem**
Two games $M$ and $G$ (played on the same number of stacks) have the same limit game if and only if their unique sets of minimal elements (with the usual partial order) are the same.

Q3: Characteristic of Reflexive Games

- The following result is somewhat technical, but it is a general result for games on any number of stacks.
- It is used to prove specific results for one and two stacks.

**Theorem**
The game $A$ on $d$ stacks is reflexive if and only if its set of moves $A$ (as a set) satisfies

$$A + A = A^c \setminus T_A$$

where $T_A$ is the set of terminal positions of the game $A$. 
Structure of Reflexive Games on One Stack

Pattern:
- period $3k-1$;
- starts at $k$
- has $k$ moves, followed by $2k-1$ non-moves.

$$M_k := \{ i \ p_k + k, \ldots, i \ p_k + (2k-1) | i = 0, 1, \ldots \}, \text{ where } p_k = 3k-1$$

Theorem
The game $M$ is reflexive iff $M = M_k$ for some $k > 0$.

Structure of Limit/Reflexive Games on Two Stacks

Classification of games according to minimal moves

1. Exactly one minimal move
   a. Not on an axis
   b. On one of the axes

2. Exactly two minimal moves
   a. No minimal move on an axis
   b. Exactly one move is on an axis
   c. Both moves are on the axes

3. Three or more minimal moves
   a. No minimal move on an axis
   b. Exactly one move is on an axis
   c. Two moves are on the axes
Example: Two Minima on Axes

Definition of Game $M_{j,k}$
Reflexivity of $M_{j,k}$

**Theorem** [Bloomfield, Dufour, Heubach, Larsson]
The game $M_{j,k}$ is reflexive.

**Corollary**
The limit game of a set $M$ equals the game $M_{j,k}$ if and only if the set of minimal elements of $M$ is $\{(j,0),(0,k)\}$.

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**Q4: How Long until Convergence?**

- We can only answer this question for games on one stack and for specific initial games

**Theorem**
For $M = \{k\}$ with $k > 1$ it takes exactly 5 iterations for the limit game to appear for the first time.

**Proof:** We explicitly derive the games $M^1$ through $M^5$.

- For games on two stacks we have very varied results from our computer explorations
Future Work

1. Investigate the structure of the limit games in the other classes for games on two stacks

2. Computer experiments for three minimal elements have produced “L-shaped” limit games, limit games with diagonal stripes, and limit games that combine the two features

Three+ Minimal Moves – None on Axis

M = {(2,9), (3,7), (4,4), (5,2), (8,1)}

Convergence after 2 steps!
Three Minimal Moves – Two on Axes

\[ M = \{(0,5), (1,1), (5,0)\} \]

Convergence after 8 steps!

Three minimal moves – two on axes

\[ M = \{(0,5), (2,2), (5,0)\} \]

Convergence after 7 steps!
Three Minimal Moves – Two on Axes

\[ M = \{(0,5), (3,3), (5,0)\} \]

Convergence after 7 steps!

Three Minimal Moves – Two on Axes

\[ M = \{(0,5), (4,4), (5,0)\} \]

Convergence after 6 steps!
Future Work

1. Investigate the structure of the limit games in the other classes for games on two stacks

2. We have observed “L-shaped” limit games, limit games with diagonal stripes, and limit games that combine the two features

3. Number of steps to convergence, or showing that it happens in a finite number of steps for all games or for games of a particular (sub-) class

Conjecture
For all subtraction games on two stacks, limit games under the misère *-operator are ultimately periodic along any line of rational slope.
References


THANK YOU!

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Slides will eventually be posted on my web site

http://web.calstatela.edu/faculty/sheubac