Nim, Wythoff and beyond - let’s play!

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Nim and Wythoff

- **Nim**: Select one of the $n$ stacks, take at least one token

- **Wythoff**: Take any number of tokens from one stack OR select the same number of tokens from both stacks
How to win???

**Question**: For a given starting position (= heights of the stacks) in a game, can we determine whether Player I or Player II has a winning strategy, that is, can make moves in such a way that s/he will win, no matter how the other player plays? (Last player to move wins)

**Goal**: Determine the set of losing positions, that is, all positions that result in a loss for the player playing from that position.

**Smaller Goal**: Say something about the structure of the losing positions.
Combinatorial Games

Definition

An *impartial combinatorial game* has the following properties:

- each player has the **same moves** available at each point in the game (as opposed to chess, where there are white and black pieces).

- **no randomness** (dice, spinners) is involved and each player has **complete information** about the game and the potential moves.
Analyzing Nim and Wythoff

Definition

A **position** in the game is denoted by $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, where $p_i \geq 0$ denotes the number of tokens in stack $i$. A position that can be reached from the current position by a legal move is called an **option**. The directed graph which has the positions as the nodes and an arrow between a position and its options is called the **game graph**.

We do not distinguish between a position and any of its rearrangements. We will use the position that is ordered in decreasing order as the representative.
Options of position \((3, 2)\) in Nim and Wythoff

\((3, 2)\) \(\sim\)

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Additional moves for Wythoff

\((3, 2)\) \(\sim\)
Options of position \((3, 2)\) in Nim and Wythoff

\[(3, 2) \sim (2, 2), (1, 2), (0, 2)\]

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Options of position (3, 2) in Nim and Wythoff

(3, 2) ~→ (2, 2), (1, 2), (0, 2)

(3, 2) ~→ (3, 1), (3, 0)

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\[(3, 2) \sim (2, 1), (1, 0)\]
Options of position $(3, 2)$ in Nim and Wythoff

$(3, 2) \rightsquigarrow (2, 2), (1, 2), (0, 2)$

$(3, 2) \rightsquigarrow (3, 1), (3, 0)$

Additional moves for Wythoff

$(3, 2) \rightsquigarrow (2, 1), (1, 0)$

Overall

$(3, 2) \rightsquigarrow (3, 1), (3, 0), (2, 2), (2, 1), (2, 0)$ for Nim

$(3, 2) \rightsquigarrow (3, 1), (3, 0), (2, 2), (2, 1), (2, 0), (1, 0)$ for Wythoff
Game graph for position \((3, 2)\) for Nim

\((3, 2)\)
Game graph for position $(3, 2)$ for Nim
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Game graph for position $(3, 2)$ for Nim
Game graph for position (3, 2) for Wythoff
Impartial Games

Definition

A position is a \( \mathcal{P} \) position for the player about to make a move if the \( \mathcal{P} \)revious player can force a win (that is, the player about to make a move is in a losing position). The position is a \( \mathcal{N} \) position if the \( \mathcal{N} \)ext player (the player about to make a move) can force a win.

For impartial games, there are only two outcome classes for any position, namely \textbf{winning position} (\( \mathcal{N} \) position) or \textbf{losing position} (\( \mathcal{P} \) position). The set of \textbf{losing positions} is denoted by \( \mathcal{L} \).
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game graph **recursively** as follows:

- **Sinks** of the game graph are always **losing** ($\mathcal{P}$) positions.
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Next we select any position (node) whose options (offsprings) are all labeled. There are two cases:
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- The position has at least one option that is a losing (**P**) position
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- All options of the position are winning ($\mathcal{N}$) positions
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- The position has at least one option that is a losing ($\mathcal{P}$) position
  $\Rightarrow$ winning position and should be labeled $\mathcal{N}$
- All options of the position are winning ($\mathcal{N}$) positions
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game graph **recursively** as follows:

- **Sinks** of the game graph are always **losing** (\(P\)) positions.

Next we select any position (node) whose **options** (offsprings) are all labeled. There are two cases:

- The position has at least one option that is a losing (\(P\)) position
  \(\Rightarrow\) **winning** position and should be labeled \(N\)

- All options of the position are winning (\(N\)) positions
  \(\Rightarrow\) **losing** position and should be labeled \(P\)
Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game graph recursively as follows:

- **Sinks** of the game graph are always losing ($\mathcal{P}$) positions.

Next we select any position (node) whose options (offsprings) are all labeled. There are two cases:

- The position has at least one option that is a losing ($\mathcal{P}$) position  
  \[ \Rightarrow \text{winning position and should be labeled } \mathcal{N} \]

- All options of the position are winning ($\mathcal{N}$) positions  
  \[ \Rightarrow \text{losing position and should be labeled } \mathcal{P} \]

The label of the starting position of the game then tells whether Player I ($\mathcal{N}$) or Player II ($\mathcal{P}$) has a winning strategy.
Is $(3, 2)$ winning or losing for Nim?
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Is (3, 2) winning or losing for Nim?
Is $(3, 2)$ winning or losing for Wythoff?
Is \((3, 2)\) winning or losing for Wythoff?
Take home lesson

- There is no legal move from a losing position to another losing position
- There is a recursive way to determine whether a position is losing or winning
- One can define a recursive function, the Grundy function, whose value is zero for a losing position, and positive for a winning position.
- Using a computer program, one can then obtain losing positions and guess a pattern for the losing positions.
An important tool

**Theorem**

Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets $A$ and $B$ with the properties:

I. every option of a position in $A$ is in $B$;

II. every position in $B$ has at least one option in $A$; and

III. the final positions are in $A$.

Then $A = \mathcal{L}$ and $B = \mathcal{W}$.
Proof strategy

- Obtain a candidate set $S$ for the set of losing positions $\mathcal{L}$
- Show that any move from a position $p \in S$ leads to a position $p' \notin S$ (I)
- Show that for every position $p \notin S$, there is a move that leads to a position $p' \in S$ (II)

Often (as is the case for Nim and Wythoff), $(0, 0, \ldots, 0)$ is the only final position and it is easy to see that (III) is satisfied.
How to win in Nim

Definition

The *digital sum* \( a \oplus b \oplus \cdots \oplus k \) of integers \( a, b, \ldots, k \) is obtained by translating their binary representation and then adding them without carry-over.
How to win in Nim

Definition

The *digital sum* $a \oplus b \oplus \cdots \oplus k$ of integers $a, b, \ldots, k$ is obtained by translating the values into their binary representation and then adding them without carry-over.

Example

The digital sum $12 \oplus 13 \oplus 7$ equals 6:

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How to win in Nim

Theorem

For the game of Nim, the set of losing positions is given by
\[ \mathcal{L} = \{ (p_1, p_2, \ldots, p_n) \mid p_1 \oplus p_2 \oplus \cdots \oplus p_n = 0 \}. \]
How to win in Wythoff

Let \( \varphi = \frac{1+\sqrt{5}}{2} \). Then the set of losing positions is given by

\[
\mathcal{L} = \{([n \cdot \varphi], [n \cdot \varphi] + n) | n \geq 0\}
\]

Elements \((a_n, b_n) \in \mathcal{L}\) can be created recursively as follows:

- For \(a_n\), find the smallest positive integer not yet used for \(a_i\) and \(b_i\), \(i < n\).
- \(b_n = a_n + n\).

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Theorem

For the game of Wythoff, for any given position \((a, b)\) there is exactly one losing position of each of the forms \((a, y), (x, b), (z, z + (b - a))\) for some \(x \geq 0, y \geq 0, \text{ and } z \geq 0\).

This structural result can be visualized as follows:
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This structural result can be visualized as follows: \((a, b) = (6, 5)\)
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This structural result can be visualized as follows: \((a, b) = (6, 5)\)

Losing positions: \((6, 10), (3, 5),\) and \((2, 1).\)
Generalization of Wythoff to $n$ stacks

Wythoff: Take any number of tokens from one stack OR select the same number of tokens from both stacks

Generalization: Take any number of tokens from one stack OR
Generalization of Wythoff to $n$ stacks

Wythoff: Take any number of tokens from one stack OR select the same number of tokens from both stacks

Generalization: Take any number of tokens from one stack OR

- take the same number of tokens from ALL stacks
Generalization of Wythoff to $n$ stacks

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  - take the same number of tokens from any TWO stacks
Generalization of Wythoff to $n$ stacks

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Generalization: Take any number of tokens from one stack OR

- take the same number of tokens from ALL stacks
- take the same number of tokens from any TWO stacks
- take the same number of tokens from any non-empty SUBSET of stacks
Generalized Wythoff on $n$ stacks

Let $B \subseteq \mathcal{P}(\{1, 2, 3, \ldots , n\})$ with the following conditions:

1. $\emptyset \notin B$
2. $\{i\} \in B$ for $i = 1, \ldots , n$.

A legal move in generalized Wythoff $\mathcal{GW}_n(B)$ on $n$ stacks induced by $B$ consists of:

- Choose a set $A \in B$
- Remove the same number of tokens from each stack whose index is in $A$

The first player who cannot move loses.
Examples

- **Nim**: Select one of the $n$ stacks, take at least one token

- **Wythoff**: Either take any number of tokens from **one** stack OR select the **same** number of tokens from both stacks
Examples

- **Nim**: Select one of the $n$ stacks, take at least one token
  \[ B = \{\{1\}, \{2\}, \ldots, \{n\}\} \]

- **Wythoff**: Either take any number of tokens from one stack OR select the same number of tokens from both stacks
Examples

- **Nim**: Select one of the $n$ stacks, take at least one token
  
  \[
  B = \{\{1\}, \{2\}, \ldots, \{n\}\}
  \]

- **Wythoff**: Either take any number of tokens from one stack OR select the same number of tokens from both stacks

  \[
  B = \{\{1\}, \{2\}, \{1, 2\}\}
  \]
\[ \vec{e}_i = \text{ith unit vector}; \quad \vec{e}_A = \sum_{i \in A} \vec{e}_i \]

**Conjecture**

In the game of generalized Wythoff \( \mathcal{GW}_n(B) \), for any position \( \vec{p} = (p_1, p_2, \ldots, p_n) \) and any \( A = \{i_1, i_2, \ldots, i_k\} \in B \), there is a unique losing position of the form \( \vec{p} + m \cdot \vec{e}_A \), where \( m \geq -\min_{i \in A} \{p_i\} \).

**Theorem**

The conjecture is true for \( |A| \leq 2 \), that is, for any given position we can find a losing position for which only one or two of the stack heights are changed.
**Example**

\[ GW_3(\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}) \] - three stacks, with play on either a single or a pair of stacks. \( \vec{p} = (11, 17, 20) \)

<table>
<thead>
<tr>
<th>( A )</th>
<th>( \vec{p} \in \mathcal{L} )</th>
<th>=</th>
<th>( \vec{p} )</th>
<th>+</th>
<th>( m \cdot \vec{e}_A )</th>
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<tr>
<td>{1}</td>
<td>( (26, 17, 20) )</td>
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<td>+</td>
<td>( 15 \cdot (1, 0, 0) )</td>
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<td>{2}</td>
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<td>( 14 \cdot (0, 1, 0) )</td>
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<td>{3}</td>
<td>( (11, 17, 36) )</td>
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<td>( (11, 17, 20) )</td>
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<td>( 16 \cdot (0, 0, 1) )</td>
</tr>
<tr>
<td>{1, 2}</td>
<td>( (19, 25, 20) )</td>
<td>=</td>
<td>( (11, 17, 20) )</td>
<td>+</td>
<td>( 8 \cdot (1, 1, 0) )</td>
</tr>
<tr>
<td>{1, 3}</td>
<td>( (1, 17, 10) )</td>
<td>=</td>
<td>( (11, 17, 20) )</td>
<td>−</td>
<td>( 10 \cdot (1, 0, 1) )</td>
</tr>
<tr>
<td>{2, 3}</td>
<td>( (11, 35, 38) )</td>
<td>=</td>
<td>( (11, 17, 20) )</td>
<td>+</td>
<td>( 18 \cdot (0, 1, 1) )</td>
</tr>
</tbody>
</table>
### Example

$$B_1 = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}; \quad B_2 = B_1 \cup \{1, 2, 3\}$$

$$\vec{p} = (11, 17, 20)$$

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\tilde{p}_1$</th>
<th>$\tilde{p}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1}$</td>
<td>(26, 17, 20)</td>
<td>(40, 17, 20)</td>
</tr>
<tr>
<td>${2}$</td>
<td>(11, 31, 20)</td>
<td>(11, 1, 20)</td>
</tr>
<tr>
<td>${3}$</td>
<td>(11, 17, 36)</td>
<td>(11, 17, 27)</td>
</tr>
<tr>
<td>${1, 2}$</td>
<td>(19, 25, 20)</td>
<td>(7, 13, 20)</td>
</tr>
<tr>
<td>${1, 3}$</td>
<td>(1, 17, 10)</td>
<td>(8, 17, 17)</td>
</tr>
<tr>
<td>${2, 3}$</td>
<td>(11, 35, 38)</td>
<td>(11, 12, 15)</td>
</tr>
<tr>
<td>${1, 2, 3}$</td>
<td>—</td>
<td>(15, 21, 24)</td>
</tr>
</tbody>
</table>
Proof for $|A| = 1$.

To show: For any position $(p_1, p_2, \ldots, p_n)$ there exists a unique position $(x, p_2, \ldots, p_n) \in \mathcal{L}$.

**Uniqueness:** Assume there are at least two positions of this form, \(\tilde{p}_1 = (x, p_2, \ldots, p_n)\) and \(\tilde{p}_2 = (y, p_2, \ldots, p_n)\), both in \(\mathcal{L}\), with \(x > y\). Then there exists a legal move from a losing position to a losing position (which is not possible) by taking \(x - y\) tokens from the first stack of \(\tilde{p}_1 = (x, p_2, \ldots, p_n)\). This is an allowed move as \(B\) always contains the singletons.
Proof for $|A| = 1$ continued.

**Existence:** Assume all positions of the form $p = (x, p_2, \ldots, p_n)$ are winning positions. Upper bound on the number of moves from $p$:

- $2^n - 1$ ways to choose the stacks to play on
- $\max_{i=2\ldots n} p_i$ different choices for number of tokens
- Let $M = (2^n - 1)(\max_{i=2\ldots n} p_i)$.

Now consider the $M + 1$ positions

$$(0, p_2, \ldots, p_n)$$
$$(1, p_2, \ldots, p_n)$$
$$\vdots$$
$$(M, p_2, \ldots, p_n)$$
Proof for $|A| = 1$ continued.

$(i, p_2, \ldots, p_n) \in \mathcal{W}$ implies that there is at least one move $t_i$ from $(i, p_2, \ldots, p_n)$ to a losing position $q_i$.

\[
\begin{align*}
(0, p_2, \ldots, p_n) + t_0 &= q_0 \in \mathcal{L} \\
(1, p_2, \ldots, p_n) + t_1 &= q_1 \in \mathcal{L} \\
&\vdots \\
(M, p_2, \ldots, p_n) + t_M &= q_M \in \mathcal{L}
\end{align*}
\]
Proof for $|A| = 1$ continued.

By the pigeon hole principle, there must be a repeated move, say $t$, yielding

$$q_i = (i, p_2, \ldots, p_n) - t = (i - t_1, p_2 - t_2, \ldots, p_n - t_n) \in \mathcal{L}$$

$$q_j = (j, p_2, \ldots, p_n) - t = (j - t_1, p_2 - t_2, \ldots, p_n - t_n) \in \mathcal{L}$$

But we already saw that this is not possible, and so there must be a losing position of the form $(x, p_2, \ldots, p_n)$. The proof easily applies to any set $A = \{i\}$.

Note: What we have proved is that from any position we can “see” a losing position in any direction parallel to one of the axes of $\mathbb{R}^n$. 
Proof for $|A| = 2$

- Proof is much more complicated
- We define the notion of a Wythoff set (a set that generalizes the properties of the set of losing positions constructed recursively for Wythoff)
- Uses a theorem about the interplay between the cardinalities of a sequence of two increasing sets and their accumulated sizes (= sums of their respective elements)
- Does not yet seem to generalize to $|A| > 2$. 
Thank You!

Slides available from

http://www.calstatela.edu/faculty/sheubac
References and Further Reading


S. Heubach, M. Dufour

Nim, Wythoff and beyond - let’s play!
References and Further Reading II


Mex

Definition

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by \( \text{mex}\{a, b, c, \ldots, k\} \).

Example

\[
\begin{align*}
\text{mex}\{1, 4, 5, 7\} &= \text{mex}\{0, 1, 2, 6\} =
\end{align*}
\]
Definition

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by \( \text{mex}\{a, b, c, \ldots, k\} \).

Example

\[
\text{mex}\{1, 4, 5, 7\} = 0 \\
\text{mex}\{0, 1, 2, 6\} = 
\]
**Definition**

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by $\text{mex}\{a, b, c, \ldots, k\}$.

**Example**

\[
\begin{align*}
\text{mex}\{1, 4, 5, 7\} &= 0 \\
\text{mex}\{0, 1, 2, 6\} &= 3
\end{align*}
\]
The Grundy Function

Definition

The Grundy function $G(p)$ of a position $p$ is defined recursively as follows:

- $G(p) = 0$ for any final position $p$.
- $G(p) = \text{mex}\{G(q) | q \text{ is an option of } p\}$.  

S. Heubach, M. Dufour  Nim, Wythoff and beyond - let’s play!
**The Grundy Function**

**Definition**

The **Grundy function** $G(p)$ of a position $p$ is defined recursively as follows:

- $G(p) = 0$ for any final position $p$.
- $G(p) = \text{mex}\{G(q)\mid q \text{ is an option of } p\}$.

**Theorem**

*For a finite impartial game, $p$ belongs to class $P$ if and only if $G(p) = 0$.***