A Generalization of the Nim and Wythoff games

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Nim and Wythoff

- **Nim:** Select one of the $n$ stacks, take at least one token

- **Wythoff:** Take any number of tokens from one stack OR select the same number of tokens from both stacks
Generalization of Wythoff to $n$ stacks

Wythoff: Take any number of tokens from one stack OR select the same number of tokens from both stacks

**Generalization:** Take any number of tokens from one stack OR

- take the same number of tokens from all stacks
- take the same number of tokens from any two stacks
- take the same number of tokens from any non-empty subset of stacks
Generalized Wythoff on $n$ stacks

Let $B \subseteq \mathcal{P}({1, 2, 3, \ldots, n})$ with the following conditions:

1. $\emptyset \notin B$
2. $\{i\} \in B$ for $i = 1, \ldots, n$.

A legal move in generalized Wythoff $GW_n(B)$ on $n$ stacks induced by $B$ consists of:

- Choose a set $A \in B$
- Remove the same number of tokens from each stack whose index is in $A$

The first player who cannot move loses.
Examples

- **Nim**: Select one of the $n$ stacks, take at least one token

- **Wythoff**: Either take any number of tokens from one stack OR select the same number of tokens from both stacks
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- **Wythoff**: Either take any number of tokens from one stack OR select the same number of tokens from both stacks
  \[ B = \{\{1\}, \{2\}, \{1,2\}\} \]
Goal

- Generalized Wythoff is a two-player impartial game
- All positions (configurations of stack heights) are either winning or losing

**Goal:** Determine the set of losing positions

**Smaller Goal:** Say something about the structure of the losing positions
Results for Wythoff

Let $\Phi = \frac{1+\sqrt{5}}{2}$. Then the set of losing positions is given by

$$\mathcal{L} = \{(\lfloor n \cdot \Phi \rfloor, \lfloor n \cdot \Phi \rfloor + n) | n \geq 0\}$$

They can be created recursively as follows:

- For $a_n$, find the smallest positive integer not yet used for $a_i$ and $b_i$, $i < n$.
- $b_n = a_n + n$. Repeat...

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
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<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>$b_n$</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>13</td>
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Theorem

For the game of Wythoff, for any given position \((a, b)\), there is exactly one a losing position of the form \((a, y), (x, b), (z, z + |b - a|)\) for some \(x \geq 0, y \geq 0, \text{ and } z \geq 0\).

This structural result can be visualized as follows:
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*For the game of Wythoff, for any given position \((a, b)\), there is exactly one losing position of the form \((a, y), (x, b), (z, z + \lvert b - a \rvert)\) for some \(x \geq 0, y \geq 0,\) and \(z \geq 0\).*

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Losing positions: \((6, 10), (3, 5), \text{ and } (2, 1)\).
\( \vec{e}_i = \text{\textit{i}th unit vector}; \ \vec{e}_A = \sum_{i \in A} \vec{e}_i \)

**Conjecture**

In the game of generalized Wythoff \( GW_n(B) \), for any position \( \vec{p} = (p_1, p_2, \ldots, p_n) \) and any \( A = \{i_1, i_2, \ldots, i_k\} \subseteq B \), there is a unique losing position of the form \( \vec{p} + m \cdot \vec{e}_A \), where \( m \geq -\min_{i \in A} \{p_i\} \).

**Theorem**

The conjecture is true for \( |A| \leq 2 \), that is, if play is either on a single stack or any pair of two stacks.
Example

\[ GW_3(\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}) - \text{three stacks, with play on either a single or a pair of stacks.} \quad \vec{p} = (11, 17, 20) \]

<table>
<thead>
<tr>
<th>(A)</th>
<th>(\vec{p})</th>
<th>(\vec{\tilde{p}})</th>
<th>(\vec{p})</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1}</td>
<td>(26, 17, 20)</td>
<td>(11, 17, 20)</td>
<td>(11, 17, 20) + 15 \cdot (1, 0, 0)</td>
</tr>
<tr>
<td>{2}</td>
<td>(11, 31, 20)</td>
<td>(11, 17, 20)</td>
<td>(11, 17, 20) + 14 \cdot (0, 1, 0)</td>
</tr>
<tr>
<td>{3}</td>
<td>(11, 17, 36)</td>
<td>(11, 17, 20)</td>
<td>(11, 17, 20) + 16 \cdot (0, 0, 1)</td>
</tr>
<tr>
<td>{1, 2}</td>
<td>(19, 25, 20)</td>
<td>(11, 17, 20)</td>
<td>(11, 17, 20) + 8 \cdot (1, 1, 0)</td>
</tr>
<tr>
<td>{1, 3}</td>
<td>(1, 17, 10)</td>
<td>(11, 17, 20)</td>
<td>(11, 17, 20) − 10 \cdot (1, 0, 1)</td>
</tr>
<tr>
<td>{2, 3}</td>
<td>(11, 35, 38)</td>
<td>(11, 17, 20)</td>
<td>(11, 17, 20) + 18 \cdot (0, 1, 1)</td>
</tr>
</tbody>
</table>
Example

\[ B_1 = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}; \quad B_2 = B_1 \cup \{1, 2, 3\} \]
\[ \vec{p} = (11, 17, 20) \]

<table>
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<tr>
<th>(A)</th>
<th>(\tilde{p}_1)</th>
<th>(\tilde{p}_2)</th>
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<tbody>
<tr>
<td>{1}</td>
<td>(26, 17, 20)</td>
<td>(40, 17, 20)</td>
</tr>
<tr>
<td>{2}</td>
<td>(11, 31, 20)</td>
<td>(11, 1, 20)</td>
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<tr>
<td>{3}</td>
<td>(11, 17, 36)</td>
<td>(11, 17, 27)</td>
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<tr>
<td>{1, 2}</td>
<td>(19, 25, 20)</td>
<td>(7, 13, 20)</td>
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<td>{1, 3}</td>
<td>(1, 17, 10)</td>
<td>(8, 17, 17)</td>
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<tr>
<td>{2, 3}</td>
<td>(11, 35, 38)</td>
<td>(11, 12, 15)</td>
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<td>{1, 2, 3}</td>
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<td>(15, 21, 24)</td>
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</table>
Proof Outline.

- For play on one stack, assuming no such position exists leads to contradiction (legal move from losing position to losing position) as there are only finitely many moves.

- For play on a pair of stacks, a somewhat different argument is needed that does not generalize to three or more stacks.
Thank You!