Graphs of Tilings

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Abstract

For a given finite set of tiles and a strip of fixed height we describe how to obtain the associated (finite directed) graph that encodes the structure of the set of all possible tilings. We then consider possible constraints on such graphs and introduce and give some preliminary results on the reverse question: Given a graph, does there exist a set of tiles and a fixed height such that the corresponding set of tilings would generate the given graph?

Keywords: Polyominoes, strip tilings, tiling graph.

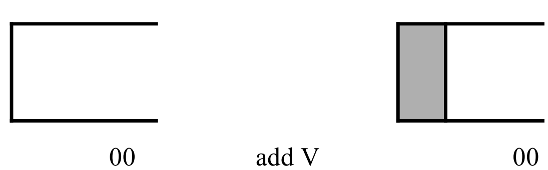
1. Introduction

The enumeration and classification of tilings has a long history. In 1953 Solomon Golomb introduced polyominoes, connected figures formed by squares placed so that each square shares a side with at least one other square (see [1], [2], and [3]). In 1965 Golomb's book on the subject, *Polyominoes: Puzzles, Patterns, Problems and Packings* [4], was published and contained many problems associated with polyominoes. One type of problem that has generated much research is the question of tiling rectangles with polyominoes. The basic question is: Given a set of polyominoes, in how many ways can one tile a KxN rectangle with those tiles? The easiest example is: In how many ways can one tile a 2xN rectangle with dominoes? The answer to the latter question is given by F_N, the Nth Fibonacci number. In a previous paper [6] the authors give formulae for the number of tilings of 2xN and 3xN rectangles with the complete set of trominoes (polyominoes of area three). Here, we are interested in representing the "structure" of these tilings.

2. Basic definitions and examples

Consider a fixed set of tiles $\mathfrak{T}$, where tiles are *not* considered equivalent under rotation. For example, if we want to tile a rectangle with dominoes we would distinguish the horizontal and vertical orientations. For trominoes there are two basic types, namely the straight (1x3) S and the L-shaped tile created by removing a 1x1 tile from a 2x2 square. However, the complete set of tromino tiles would have six elements (the two orientations of the S, and the four orientations of the L). Now consider a fixed number K for the height of the rectangle. We can put down tiles one by one in lexicographic order (top to bottom, then left to right). This means that we can unambiguously describe a tiling as an ordered sequence of tiles. This approach has been used in many different settings: Merlini et al. [7] used it to generate a grammar and from it the generating function for the number of tilings; Stanley [8] refers to this approach as a transfer-matrix method, and Graham et al. [5] use this technique as an example of a finite state machine. These are all powerful and general tools but they can also be a bit opaque. We would like to focus on the graph representation of the tiling to easily visualize its structure. Namely, each time a tile is placed down in lexicographic order there are a finite number of choices for the next tile. Sometimes there are no *allowable* choices, i.e., ways to place a tile in such a way that it does not overlap with existing tiles in the sequence. Notice that the only information that impacts whether a tile can be added is the *end configuration* or *end pattern*, which consists of those squares that have *free edges*. A tile has a free edge if there is no tile horizontally adjacent to one or more of the squares covered by the tile. Since there are finitely many tiles, there are finitely many end configurations. Each of these configurations becomes a node in the *tiling graph*. Two vertices u and v are connected by a directed edge if there is a tile that can be placed in lexicographic order such that the configuration u is transformed into the configuration v. The tiling graph encodes all the information on how tiles can be sequentially fit together in a rectangular strip of height K.

Example: Let $\mathfrak{T}$ be the set of dominoes, with H denoting the horizontal orientation, and V denoting the vertical orientation of the domino tile. What are the different ways to tile the 2xN strips with dominoes? Figure 1 shows the possible configurations and the transformations between them. (We will explain the numeric encodings of the end configurations below.)



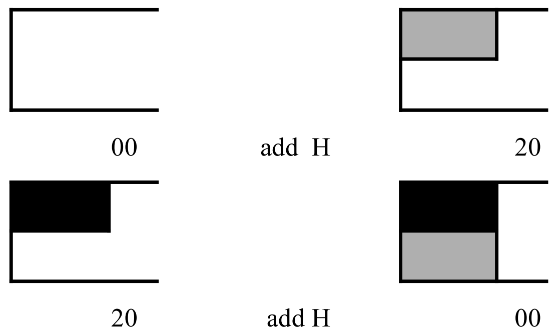


Figure 1. Domino tilings

Note that there are only two possible configurations, 00 and 20. The corresponding tiling graph is shown in Figure 2, where the edges are labeled according to the tile that transforms one configuration into the other.

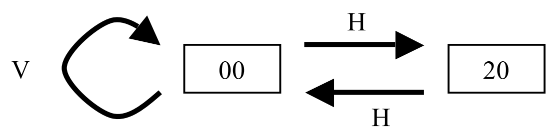


Figure 2. Tiling graph for domino tilings

We will now describe the encoding of the end configurations, which are important because they determine the restrictions on the tiles that can be added at the next step in the sequence (if any). For a rectangle of fixed height *K*, we will assign *K* numbers to uniquely describe the end pattern. First we identify the (vertical) free edge furthest to the left (assuming that the lexicographic ordering of the tiles is proceeding from left to right). This free edge furthest to the left will serve as the "origin" or zero level. We assign the binary values 2^m to the *m*-th cell to the right of the zero level, and then add the values for the occupied cells to the right of the zero level in each row. The resulting *K* values uniquely describe the end configuration, even if there are uncovered cells between the zero level and the free edge. This can occur for example when tiling with trominoes, and is shown in the tiling on the right in Figure 3. We will use the *K* values for each configuration as the label for the corresponding vertex in the tiling graph. The tiling on the left has label 701, and the one on the right has label 320.

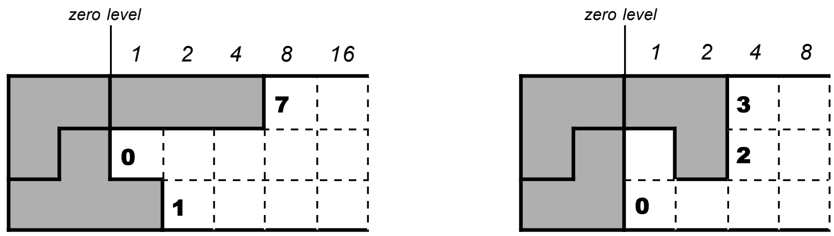


Figure 3. The end configurations 701 and 320

Note that this encoding gives an upper bound on the number of end patterns. If d is the maximal dimension of the tiles in $\mathfrak{T}$ and K is the height of the rectangular strip, then the number of configurations is bounded above by 2^{d^K}. Of course this bound generally far exceeds the actual number of end configurations since many of the patterns cannot arise.

The algorithm above for describing the end patterns works for all types of tilings. However, many examples have relatively simple tiles, and it is often easier to see the structure by modifying the algorithm slightly. Instead of using binary encoding, one can instead count the squares to the right of the zero level. Then the tiling on the left of Figure 3 would be encoded by 301, rather than by 701. For the tiling on the right, we need to make a specific rule to account for the exceptions that occur when there are empty cells between the zero level and the free edge on the right, so that such a configuration can be distinguished from the one where the tile covers all cells between the zero level and the free edge. Instead of labeling the tiling on the right in Figure 3 by 220, we will label it as 240 in the tiling graphs for the domino tilings. We will use the simplified code in the tiling graphs shown in Figures 4 and 5.

3. Why look at tiling graphs?

The tiling graph is a representation of the structure of the tiling sequence of a particular set of tiles and a rectangular strip. As such it encodes the information we need to answer questions such as "How many ways can one tile a 2xN strip with trominoes?" Tools and techniques such as finite state machines and regular grammars can encode the same information and provide answers to the same questions, but the tiling graph gives a visual representation, and we can use the machinery and techniques of graph theory.

Example: Figure 4 shows the tiling graph for rectangular strips of height 3 tiled by trominoes. The edges are labeled with the shape of tromino that transforms the end states. Recall that tiles are not allowed to be rotated, so each orientation can induce different edges in the graph. Figure 5 shows a simplified version of the same tiling graph, where the edge labels have been replaced by dashed lines for the straight pieces and solid lines for the L pieces.

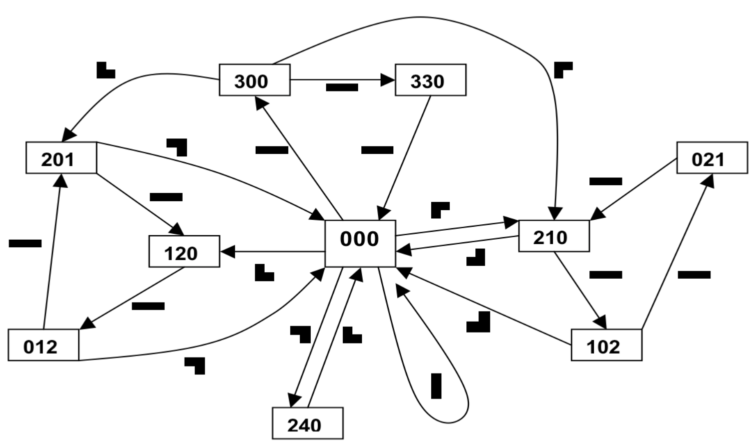


Figure 4. The tiling graph for the tromino tilings of size 3x*N*

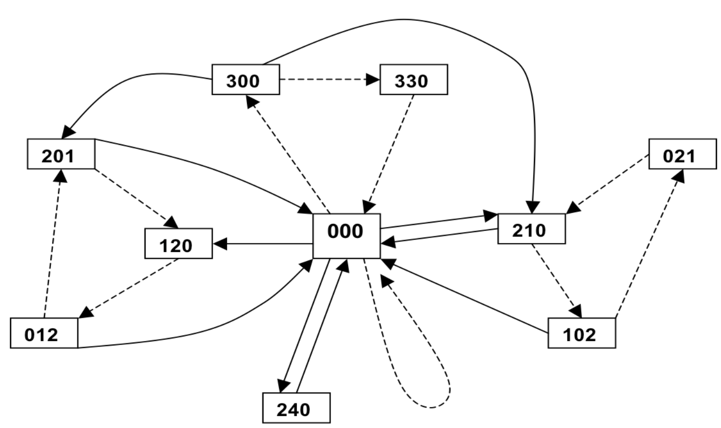


Figure 5. Simplified tiling graph for the tromino tilings of size 3x*N*

In this example, by using the tiling graph representation we can "see" some of the structure of the set of tilings. The horizontal straight pieces form three sets of triangles in the graph. The vertical straight piece only shows up as a loop based at the "origin". Except for the 2-cycle containing vertex "240", each of the four orientations appears exactly twice in the graph.

One immediate observation is that there is a one-to-one correspondence between loops in the tiling graph that start and end at "0…0" (the end that has all free edges at the zero level) and complete tilings. This allows for a procedure to enumerate tilings based on the tiling graph since one can count the number of paths from one vertex to another by looking at appropriate entries in powers of the adjacency matrix.

Example: Figure 6 shows the tiling graph for rectangular stips of height two tiled with trominoes.

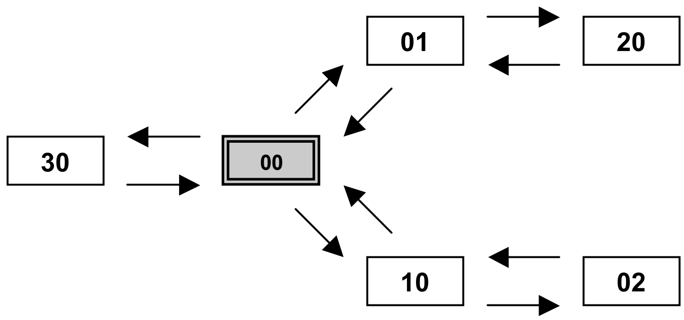


Figure 6. Tiling graph for the tromino tilings of size 2x*N*

For this tiling graph, the adjacency matrix is given by (with vertices listed in the order 00, 30, 01, 10, 02, and 20):

$$\mathbf{A} = \begin{pmatrix} 0 & S & L & L & 0 & 0 \\ S & 0 & 0 & 0 & 0 & 0 \\ L & 0 & 0 & 0 & 0 & S \\ L & 0 & 0 & 0 & S & 0 \\ 0 & 0 & 0 & S & 0 & 0 \\ 0 & 0 & S & 0 & 0 & 0 \end{pmatrix}$$

If we just want to count paths and are not interested in the types of tiles, then we can replace S and L in the adjacency matrix with 1s. Then the (1,1) entry of the m-th power of this matrix will give the number of tilings that contain exactly m tiles. However, we can get much more information by looking at the sequence of polynomials $p_n(L,S) = \left(\mathsf{A}^n\right)_{1,1}$. Note that these polynomials count the labeled paths, so that the "value" of a path is the product of the labels. Here are the first few polynomials.

$$p_2 = 2L^2 + S^2$$
$$p_4 = 4L^4 + 6L^2S^2 + S^4$$
$$p_6 = 8L^6 + 20L^4S^2 + 12L^2S^4 + S^6.$$

Since the area of all tiles is three, the polynomials for odd values of n are all zero, and p_n gives us information about tilings of the $2\times(3n/2)$ rectangle. We obtain that there are two tilings containing exactly two L tiles, and one tiling consisting of exactly two S tiles. Also, there are 20 tilings of the 2x9 rectangle with exactly four L tiles and two S tiles.

Adding the coefficients in the polynomial p_n gives the total number of tilings that consist of exactly n tiles. Thus, there are a total of 41 tilings of the 2x9 rectangle. Adding the coefficients multiplied by the power of the variable gives the number of occurrences of a specific type of tile in all the tilings of a given size, where we do not distinguish between the orientations. (If that level of detail is desired, then the values in the adjacency matrix can be labeled L_1, L_2, L_3, L_4, S_1, and S_2.)

Let $T(2,3n)$ denote the number of tromino tilings of the $2\times(3n)$ rectangle, $T_S(2,3n)$ and $T_L(2,3n)$ the number of S and L tiles in all tromino tilings of the $2\times(3n)$ rectangle, respectively, and let $p_n(L,S) = \sum_{k=0}^{n} c_{n,k}L^kS^{n-k}$. From the discussion above we can obtain the results given in Theorems 3.1 and 3.2 of [6], albeit not in explicit form.

Theorem 1.

1) $T(2,3n) = p_{2n}(1,1)$.
2) $T_S(2,3n) = \sum_{k=0}^{2n} c_{2n,k}\cdot(2n-k)$ and $T_L(2,3n) = \sum_{k=0}^{2n} c_{2n,k}\cdot k$.

4. Which graphs are tiling graphs?

From the discussion in Section 2 it is clear that we can create a tiling graph for each set of tiles $\mathfrak{T}$ and height K of the rectangular strip (even though it becomes hard to do by hand if the number of tiles and the height of the rectangle become larger). Thus it is natural to ask the following questions:

1. Can any graph be a tiling graph?
2. What are the necessary and/or sufficient properties for a graph to be a tiling graph?
3. If a graph is a tiling graph, can we determine the set of tiles $\mathfrak{T}$ and the height K of the rectangular strip that has this tiling graph?
4. Is the set of tiles unique up to some kind of equivalence? More specifically, is it possible to have two sets of tiles generate the same tiling graph, but the two sets of tiles cannot be put in one-to-one correspondence in any natural way?

We have obtained a few preliminary results with regard to Question 2.

Theorem 2. A tiling graph must be connected and there must be a special vertex such that every other vertex in the graph lies on a circuit that contains the special vertex (the 00…00 vertex).

Proof: It is rather obvious that the tiling graph must be connected, as an isolated vertex would correspond to an end configuration that can never be created, and can also not create another end configuration. The same argument applies to a graph that has more than one component. To prove existence of a special vertex, note that a proper tiling must have a vertical edge across the height of the strip, both at the beginning and at the end. Thus every proper tiling corresponds to a path from a vertex representing the straight edge to itself. Since every possible configuration is contained in some tiling, each vertex must be on a circuit that contains also the special vertex. ■

We can also answer Question 3 for some special types of graphs.

Theorem 3. The (directed) cycle C_n and the complete (directed) graph K_n on n vertices can be obtained as tiling graphs.

Proof: We will tile strips of height n. For both C_n and K_n, there are exactly n end configurations, shown in Figure 7. We will label them as 0, 1, 2, … n-1 for simplicity of notation.

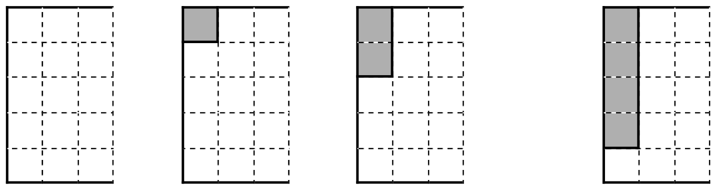


Figure 7. End configurations

For C_n, where edges are of the form (*i*, *i*+1), the set of tiles $\mathfrak{T}$ consists of a single tile, the 1x1 square. For K_n, we will need a larger set of tiles. Let V_k denote the vertically oriented 1x*k* tile. Let $S_{k,j}$ denote the "S"-shaped tile formed by taking a 3x*n* rectangle and removing V_k from the upper left corner and V_{n-j} from the lower right corner. Figure 8 shows some of these tiles for *n* = 5.

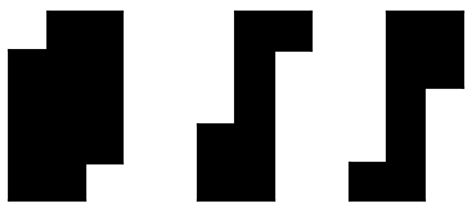


Figure 8. The tiles $S_{1,4}$, $S_{3,1}$, **and** $S_{4,2}$

Let $\mathfrak{T} = \{V_1, V_2 \ldots V_{n-1}\} \cup \{S_{k,j} | 1 \le k \le n-1 \text{ and } 1 \le j \le k-1\}$. This set of tiles creates the following edges:

- From vertex 0 to any non-zero vertex *p*: the edge corresponds to adding V_p;
- From any non-zero vertex *p* to vertex 0: the edge corresponds to adding V_{n-p};
- For vertices $p > 0$ and $q > 0$ with $p < q$: the edge from vertex *p* to vertex *q* corresponds to adding V_{q-p}; the edge from vertex *q* to vertex *p* corresponds to adding $S_{q,p}$.

Thus for any vertices *p* and *q* there is a unique edge from *p* to *q*, hence the tiling graph is the complete di-graph on *n* vertices. ■

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