Factoring Forms

Gary Brookfield

Abstract. We provide necessary and sufficient conditions for the complete reducibility of ternary forms of degree three. Curiously, this result was well-known in the 19th century, but then forgotten.

1. INTRODUCTION. No doubt, you can multiply

$$(x - y + 2)(x^2 - 2y^2 - 4y - 2) \tag{1}$$

to get

$$x^3 + 2y^3 - x^2y - 2xy^2 + 2x^2 - 4 - 4xy - 2x - 6y. \tag{2}$$

But can you start with (2) and derive its factorization (1)? And, if so, would you notice that further factorization is possible because

$$x^2 - 2y^2 - 4y - 2 = \left(x - \sqrt{2}(1 + y)\right)\left(x + \sqrt{2}(1 + y)\right). \tag{3}$$

Or consider the polynomial

$$x^3 + y^3 - 3x^2y - 3y^2 - 3xy - 3x + 1. \tag{4}$$

Can you tell if it factors at all? If it does, can you find its factorization? The answers appear later in this article. Hint: The polynomial does, in fact, factor (nontrivially), and the coefficients of the factors involve $\cos 20^\circ$.

Research into the factorization of polynomials with two and more variables flourished in the 19th century as part of the theory of invariants [**8, 14**]. Most recent research has been focused on the algorithmic aspects of the problem [**12**]. As a consequence, modern computer algebra systems can quickly factor polynomials such as (2) and (4) (at least if given the right extension field of $\mathbb{Q}$). But these algorithms answer factorization problems one polynomial at a time. The goal of this article is to point out that, for some general questions about the factorization of polynomials of low degree, the answers were found 150 years ago—and then forgotten.

2. FORMS. One lesson from the theory of invariants is that factorization is best discussed in terms of forms rather than polynomials. A **form** is simply a **homogeneous** polynomial, that is, a polynomial in variables $x_1, x_2, \ldots, x_n$ in which each term has the same total degree. For example,

$$x_1 - x_2 \qquad x_1^2 - 3x_1x_2 + 3x_3^2 \qquad x_1^3 + x_2^3 - 3x_3^3 + x_1x_2x_3 - x_2^2x_3 \tag{5}$$

are forms of degree one, two, and three, respectively. In other words, these are linear, quadratic, and cubic forms. Forms in two variables are called binary. Forms in three variables are called ternary. So the rightmost expression in (5) is a ternary cubic form.

http://dx.doi.org/10.4169/amer.math.monthly.123.4.347
MSC: Primary 12D05

We lose no generality by restricting our attention to forms. If we are interested, for example, in the factorization of (2), we first have to "homogenize" it. We replace x by x_1 and y by x_2, and then we multiply each term by a sufficiently high power of a third variable x_3 so that the result is a form. Applying this process to (2), we get the ternary cubic form

$$x_1^3 + 2x_2^3 - x_1^2x_2 - 2x_1x_2^2 + 2x_1^2x_3 - 4x_3^3 - 4x_1x_2x_3 - 2x_1x_3^2 - 6x_2x_3^2. \qquad (6)$$

To "dehomogenize" this form, we just set x_1 to x, x_2 to y, and x_3 to 1. It is not hard to see that any factorization of (2) will give a factorization of (6) and vice versa. For example, the factorization of (2) in (1) can be homogenized to give a factorization of (6):

$$(x_1 - x_2 + 2x_3)(x_1^2 - 2x_2^2 - 4x_2x_3 - 2x_3^2).$$

It is no surprise that products of forms are forms, but it is not quite so obvious that factors of forms are forms. Specifically, if f, g and h are polynomials such that $f = gh$, then f is a form if and only if g and h are forms. The proof of this fact is not hard—one needs to pay attention to the terms of lowest and highest degree in each polynomial. Consequently, when looking for a factorization of a form, we need only consider factors that are themselves forms.

A form is **reducible** if it can be written as a product of two or more forms of degree one or higher. A form is **completely reducible** if it can be written as a product of two or more linear forms. For quadratic forms, there is no difference between reducible and completely reducible. As we have seen in (3), a form may have rational coefficients but its factors have irrational coefficients. This is a reflection of the fact that the reducibility of a form depends on what we allow for the coefficients of the factors. To simplify our discussion, we will allow coefficients from the set of complex numbers, $\mathbb{C}$.

Any binary form is completely reducible over $\mathbb{C}$. For example, consider $3x_1^3 + x_1x_2^2 - 5x_2^3$. Dehomogenizing by setting x_1 to x and x_2 to 1 gives $3x^3 + x - 5$. By the fundamental theorem of algebra, this univariate polynomial can be written as

$$3x^3 + x - 5 = 3(x - \alpha)(x - \beta)(x - \gamma)$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are its zeros. Homogenizing this equation gives

$$3x_1^3 + x_1x_2^2 - 3x_2^3 = (3x_1 - 3\alpha x_2)(x_2 - \beta x_2)(x_1 - \gamma x_2),$$

a product of linear forms. This argument generalizes easily to arbitrary binary forms.

In contrast, very few ternary forms are completely reducible, even over $\mathbb{C}$. As we will see later, none of

$$x_1^2 + x_2^2 + x_3^2, \qquad x_1^2 + x_2x_3, \qquad x_1^3 + x_2^3 + x_3^3, \qquad x_1^3 + x_1x_2x_3, \qquad (7)$$

is completely reducible. Caution: $x_1^3 + x_1x_2x_3 = x_1(x_1^2 + x_2x_3)$ is reducible but is not completely reducible because $x_1^2 + x_2x_3$ does not factor further.

Since the question of the reducibility of ternary forms is the obvious next step after the binary case is resolved, it is surprising that this issue doesn't get more attention. We will show that there is a simple test (Theorem 1) for the complete reducibility of ternary forms of degree two and three. To understand this theorem, we need one more concept—the Hessian.

If f is a ternary form, then the **Hessian** of f is defined by

$$\mathcal{H} = \mathcal{H}(f) = \begin{vmatrix} \partial_{11}^2 f & \partial_{12}^2 f & \partial_{13}^2 f \\ \partial_{12}^2 f & \partial_{22}^2 f & \partial_{23}^2 f \\ \partial_{13}^2 f & \partial_{23}^2 f & \partial_{33}^2 f \end{vmatrix}$$

where, for convenience, we write $\partial_{ij}^2 f = \dfrac{\partial^2 f}{dx_i\, dx_j}$ for $i, j \in \{1, 2, 3\}$. If f has degree 2, for example,

$$f = f_{11}x_1^2 + f_{22}x_2^2 + f_{33}x_3^2 + f_{12}x_1x_2 + f_{13}x_1x_3 + f_{23}x_2x_3 \tag{8}$$

with $f_{11}, f_{22}, f_{33}, f_{12}, f_{13}, f_{23} \in \mathbb{C}$, then

$$\begin{aligned} \mathcal{H} &= \begin{vmatrix} 2f_{11} & f_{12} & f_{13} \\ f_{12} & 2f_{22} & f_{23} \\ f_{13} & f_{23} & 2f_{33} \end{vmatrix} \\ &= 2(4f_{11}f_{22}f_{33} + f_{12}f_{13}f_{23} - f_{13}^2 f_{22} - f_{11}f_{23}^2 - f_{12}^2 f_{33}), \end{aligned} \tag{9}$$

and so the Hessian of f is a constant.

If f is a cubic form, then so is its Hessian. Specifically, if

$$\begin{aligned} f = {} & f_{111}x_1^3 + f_{112}x_1^2x_2 + f_{122}x_1x_2^2 + f_{222}x_2^3 + f_{113}x_1^2x_3 \\ & + f_{123}x_1x_2x_3 + f_{223}x_2^2x_3 + f_{133}x_1x_3^2 + f_{233}x_2x_3^2 + f_{333}x_3^3, \end{aligned} \tag{10}$$

with $f_{111}, f_{112}, f_{122}, f_{222}, f_{113}, f_{123}, f_{223}, f_{133}, f_{233}, f_{333} \in \mathbb{C}$, then the Hessian of f can be written as

$$\begin{aligned} \mathcal{H} = {} & \mathcal{H}_{111}\, x_1^3 + \mathcal{H}_{112}\, x_1^2x_2 + \mathcal{H}_{122}\, x_1x_2^2 + \mathcal{H}_{222}\, x_2^3 + \mathcal{H}_{113}\, x_1^2x_3 \\ & + \mathcal{H}_{123}\, x_1x_2x_3 + \mathcal{H}_{223}\, x_2^2x_3 + \mathcal{H}_{133}\, x_1x_3^2 + \mathcal{H}_{233}\, x_2x_3^2 + \mathcal{H}_{333}\, x_3^3. \end{aligned} \tag{11}$$

The expressions for the coefficients of $\mathcal{H}$ are bulky so we list only a few that are most relevant for later calculations:

$$\begin{aligned} \mathcal{H}_{111} &= 24f_{111}f_{122}f_{133} + 8f_{112}f_{113}f_{123} - 8f_{113}^2 f_{122} - 8f_{112}^2 f_{133} - 6f_{123}^2 f_{111} \\ \mathcal{H}_{222} &= 24f_{222}f_{112}f_{233} + 8f_{122}f_{223}f_{123} - 8f_{223}^2 f_{112} - 8f_{122}^2 f_{233} - 6f_{123}^2 f_{222} \\ \mathcal{H}_{112} &= 72f_{111}f_{133}f_{222} + 24f_{111}f_{122}f_{233} - 24f_{113}^2 f_{222} - 24f_{111}f_{123}f_{223} \\ &\qquad + 16f_{112}f_{113}f_{223} - 8f_{112}^2 f_{233} - 8f_{112}f_{122}f_{133} + 2f_{112}f_{123}^2 \\ \mathcal{H}_{122} &= 72f_{111}f_{222}f_{233} + 24f_{112}f_{133}f_{222} - 24f_{223}^2 f_{111} - 24f_{113}f_{123}f_{222} \\ &\qquad + 16f_{113}f_{122}f_{223} - 8f_{122}^2 f_{133} - 8f_{112}f_{122}f_{233} + 2f_{122}f_{123}^2 \\ \mathcal{H}_{123} &= 216f_{111}f_{222}f_{333} + 24f_{112}f_{133}f_{223} + 24f_{113}f_{122}f_{233} \\ &\qquad - 24f_{111}f_{223}f_{233} - 24f_{112}f_{122}f_{333} - 24f_{113}f_{133}f_{222} \\ &\qquad - 8f_{113}f_{123}f_{223} - 8f_{112}f_{123}f_{233} - 8f_{122}f_{123}f_{133} + 2f_{123}^3. \end{aligned} \tag{12}$$

For example, with f as in (6) we get

$$\mathcal{H} = \begin{vmatrix} 6x_1 - 2x_2 + 4x_3 & -2x_1 - 4x_2 - 4x_3 & 4x_1 - 4x_2 - 4x_3 \\ -2x_1 - 4x_2 - 4x_3 & -4x_1 + 12x_2 & -4x_1 - 12x_3 \\ 4x_1 - 4x_2 - 4x_3 & -4x_1 - 12x_3 & -4x_1 - 12x_2 - 24x_3 \end{vmatrix}$$
$$= 144 f.$$

According to the main theorem of this article that follows, the fact that $\mathcal{H}$ is a multiple of f in this special case tells us that f is completely reducible.

Theorem 1. *Let f be a ternary form with Hessian $\mathcal{H}$.*

1. *If f has degree two, then f is completely reducible if and only if $\mathcal{H} = 0$.*
2. *If f has degree three, then f is completely reducible if and only if $\mathcal{H} = \lambda f$ for some $\lambda \in \mathbb{C}$.*

The claim in this theorem about quadratic forms is well-known, though it is frequently expressed in different ways by different authors. The recent article by Kronenthal and Lazebnick [**13**] provides a guide to this result (and its generalizations) in the literature.

The claim about cubic forms first appears in a paper by Aronhold [**1**, p. 145] in 1849 where it is presented as a consequence of results about ternary forms discovered a few years earlier by Hesse [**10, 11**] (after whom the Hessian gets its name). Later 19th century mathematicians extended the claim in various ways [**2, 3, 7, 9, 15, 16, 18**], but, after 1900, the claim seems to have disappeared from the literature.

In one direction, Theorem 1 is easy to prove. If, for example, f is a completely reducible ternary cubic form, then $f = abc$ where $a = a_1x_1 + a_2x_2 + a_3x_3$, $b = b_1x_1 + b_2x_2 + b_3x_3$, and $c = c_1x_1 + c_2x_2 + c_3x_3$ are linear forms. A straightforward calculation of the Hessian of f gives

$$\mathcal{H}(f) = 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 abc = 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 f \qquad (13)$$

and so the Hessian is a multiple of f as claimed. If f is a completely reducible ternary quadratic form, then a similar calculation gives $\mathcal{H}(f) = 0$.

It is much harder to show that the conditions on the Hessian in Theorem 1 imply that f is completely reducible, so we postpone the rest of the proof until later.

For a first application, we calculate the Hessians of the forms in (7):

$$\mathcal{H}(x_1^2 + x_2^2 + x_3^2) = 8 \qquad \mathcal{H}(x_1^2 + x_2x_3) = -2$$
$$\mathcal{H}(x_1^3 + x_2^3 + x_3^3) = 216\, x_1x_2x_3 \qquad \mathcal{H}(x_1^3 + x_1x_2x_3) = -6\, x_1^3 + 2\, x_1x_2x_3.$$

So, according to Theorem 1, none of these forms is completely reducible.

3. FINDING THE FACTORS. Theorem 1 might make it clear when a form is completely reducible, but it does not help find the factorization. For that, we need another definition. If f is a ternary form, then the **gradient** of f is the vector of partial derivatives:

$$\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f)$$

where $\partial_i f = \dfrac{\partial f}{\partial x_i}$ for $i = 1, 2, 3$.

Suppose now that f is a completely reducible ternary cubic form and we want to find linear forms a, b and c such that $f = abc$. By the product rule, the gradient of f is

$$\nabla f = \nabla(abc) = ab\,\nabla c + ac\,\nabla b + bc\,\nabla a. \tag{14}$$

Suppose further that we have found some nonzero $u \in \mathbb{C}^3$ such that $f(u) = 0$ (that is, u is a zero of f). Since $f(u) = a(u)b(u)c(u)$, we have $a(u) = 0$, $b(0) = 0$ or $c(u) = 0$. Without loss of generality, suppose that $a(u) = 0$. Note that, if $a = a_1x_1 + a_2x_2 + a_3x_3$ with $a_1, a_2, a_3 \in \mathbb{C}$, then $\nabla a = (a_1, a_2, a_3)$ is just the vector of the coefficients of a. From (14), the gradient of f evaluated at u is

$$\nabla f(u) = b(u)c(u)\nabla a(u) = b(u)c(u)(a_1, a_2, a_3).$$

We are lucky if $\nabla f(u) \neq 0$ (or equivalently, $b(u)c(u) \neq 0$) since then (a_1, a_2, a_3) is a multiple of $\nabla f(u)$. Thus, the gradient of f at u determines the coefficients of a linear factor of f. Indeed,

$$(\nabla f(u)) \cdot (x_1, x_2, x_3) = b(u)c(u)(a_1, a_2, a_3) \cdot (x_1, x_2, x_3) = b(u)c(u)\,a \tag{15}$$

is a linear factor of f.

What if we are unlucky? What happens if $\nabla f(u) = 0$ whenever $f(u) = 0$? The reader is encouraged to show that this happens only when $f = a^3$ for some linear form a and that, in this case, the coefficients of a can be found by evaluating ∇f at any point $u \in \mathbb{C}^3$ such that $a(u) \neq 0$.

To make this concrete, let us find the factorization of f as in (6). As already noted, f is completely reducible. To find the factorization of f, we first look for a zero of f. For example, we can arbitrarily set $x_2 = 1$ and $x_3 = 0$ giving an equation in x_1 to solve: $f(x_1, 1, 0) = x_1^3 - x_1^2 - 2x_1 + 2 = 0$. Since $x_1^3 - x_1^2 - 2x_1 + 2 = (x_1 - 1)(x_1^2 - 2)$ we find, in fact, three zeros of f, namely, $u_1 = (1, 1, 0)$, $u_2 = (\sqrt{2}, 1, 0)$ and $u_3 = (-\sqrt{2}, 1, 0)$. The gradient of f is

$$\begin{aligned}\nabla f = (&3x_1^2 - 2x_1x_2 - 2x_2^2 + 4x_1x_3 - 4x_2x_3 - 2x_3^2,\\ &-x_1^2 - 4x_1x_2 + 6x_2^2 - 4x_1x_3 - 6x_3^2,\\ &2x_1^2 - 4x_1x_2 - 4x_1x_3 - 12x_2x_3 - 12x_3^2)\end{aligned}$$

and so

$$\begin{aligned}\nabla f(u_1) &= (-1, 1, -2)\\ \nabla f(u_2) &= (4 - 2\sqrt{2})(1, -\sqrt{2}, -\sqrt{2})\\ \nabla f(u_3) &= (4 + 2\sqrt{2})(1, \sqrt{2}, \sqrt{2}).\end{aligned}$$

From (15), we get three linear factors of f (up to scalar multiples):

$$a = -x_1 + x_2 - 2x_3$$
$$b = x_1 - \sqrt{2}\,x_2 - \sqrt{2}\,x_3$$
$$c = x_1 + \sqrt{2}\,x_2 + \sqrt{2}\,x_3.$$

The product of these linear factors must be a scalar multiple of f, and, in fact, we have $f = -abc$.

4. EXAMPLES. The nice thing about Theorem 1 is that it can be used to test the reducibility of entire families of forms at the same time—as the following examples show.

Example 1. Suppose that

$$f = a_1\, x_1^3 + a_2\, x_2^3 + a_3\, x_3^3 - 3m\, x_1x_2x_3$$

for some $a_1, a_2, a_3, m \in \mathbb{C}$. Then the Hessian of f is

$$\begin{aligned}\mathcal{H} &= -54\left(m^2(a_1x_1^3 + a_2x_2^3 + a_3x_3^3) + (m^3 - 4a_1a_2a_3)\,x_1x_2x_3\right)\\ &= -54m^2 f + 216(a_1a_2a_3 - m^3)x_1x_2x_3.\end{aligned}$$

By Theorem 1, f is completely reducible if and only if $(a_1a_2a_3 - m^3)x_1x_2x_3$ is a multiple of f. This can happen in two ways: Either f is a multiple of $x_1x_2x_3$, or $a_1a_2a_3 - m^3 = 0$. In the first case, the factorization of f is obvious.

How does f factor if $a_1a_2a_3 = m^3$? Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ satisfy $\alpha_1^3 = a_1$, $\alpha_2^3 = a_2$, and $\alpha_3^3 = a_3$. Since $(\alpha_1\alpha_2\alpha_3)^3 = m^3$, it is possible to choose these cube roots so that $\alpha_1\alpha_2\alpha_3 = m$. Then f factors as

$$(\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3)(\alpha_1x_1 + \alpha_2\omega x_2 + \alpha_3\omega^2x_3)(\alpha_1x_1 + \alpha_2\omega^2x_2 + \alpha_3\omega x_3),$$

where $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$, a cube root of 1. Confirming this factorization is just a calculation using $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$. One special case worth noting is $a_1 = a_2 = a_3 = m = 1$, which gives the factorization

$$\begin{aligned}&x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3\\ &\qquad = (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2x_3)(x_1 + \omega^2x_2 + \omega x_3).\end{aligned} \tag{16}$$

Example 2. Suppose that L_1 and L_2 are ternary linear forms and f is the ternary form

$$f = c_{111}\, L_1^3 + c_{112}\, L_1^2L_2 + c_{122}\, L_1L_2^2 + c_{222}\, L_2^3 \tag{17}$$

with $c_{111}, c_{112}, c_{122}, c_{222} \in \mathbb{C}$. Then, even without Theorem 1, it is clear that f is completely reducible. Indeed any factorization of the binary form

$$\begin{aligned}&c_{111}\, x_1^3 + c_{112}\, x_1^2x_2 + c_{122}\, x_1x_2^2 + c_{222}\, x_2^3\\ &\qquad = (a_1x_1 + a_2x_2)(b_1x_1 + b_2x_2)(c_1x_1 + c_2x_2)\end{aligned}$$

with $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{C}$ gives a factorization of f as

$$f = (a_1 L_1 + a_2 L_2)(b_1 L_1 + b_2 L_2)(c_1 L_1 + c_2 L_2).$$

Because of Theorem 1, we expect that the Hessian of f is a multiple of f. In fact, since the set of factors of f is linearly dependent, (13) implies that $\mathcal{H}(f) = 0$.

One notable circumstance in which (17) holds is when f is **translation invariant**, that is, f remains unchanged by a substitution of the form $x_1 \mapsto x_1 + t$, $x_2 \mapsto x_2 + t$, and $x_3 \mapsto x_3 + t$ for all $t \in \mathbb{C}$. For example, any polynomial in the linear forms $x_1 - x_2$, $x_2 - x_3$, and $x_3 - x_1$ is translation invariant. Since $(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_1) = 0$, any polynomial in the three linear forms can be written as a polynomial in any two of them. Thus, any translation invariant ternary cubic form f can be written as

$$\begin{aligned} &c_{111}\,(x_1 - x_2)^3 + c_{112}\,(x_1 - x_2)^2(x_2 - x_3) \\ &\qquad + c_{122}\,(x_1 - x_2)(x_2 - x_3)^2 + c_{222}\,(x_2 - x_3)^3 \end{aligned}$$

and so, by the above argument, is completely reducible.

Example 3. Suppose that

$$\begin{aligned} f = a&(2x_1 - x_2 - x_3)(2x_2 - x_3 - x_1)(2x_3 - x_1 - x_2) \\ &+ b(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \end{aligned}$$

for some $a, b \in \mathbb{C}$. Then f is translation invariant and so, from the previous example, the Hessian of f is zero and f is completely reducible for any a and b. The factorization can be found as suggested in Example 2, but there is a much nicer way.

There are two cases.

Case 1: Suppose that $27a^2 + b^2 = 0$. We can assume that $a \neq 0$ since otherwise $f = 0$. Then some calculation shows that

$$108a^2 f = \big(6a\,x_1 - (3a + b)\,x_2 - (3a - b)\,x_3\big)^3,$$

showing that f is completely reducible.

Case 2: Suppose that $27a^2 + b^2 \neq 0$. Let s_1, s_2, and s_3 be the zeros of

$$G = x^3 - (27a^2 + b^2)(x + 2a). \tag{18}$$

Matching coefficients in $G = (x - s_1)(x - s_2)(x - s_3)$ gives

$$\begin{aligned} s_1 + s_2 + s_3 &= 0 \\ s_1 s_2 + s_1 s_3 + s_2 s_3 &= -(27a^2 + b^2) \\ s_1 s_2 s_3 &= 2a(27a^2 + b^2). \end{aligned} \tag{19}$$

These equations imply

$$4b^2(27a^2 + b^2)^2 = (s_1 - s_2)^2(s_2 - s_3)^2(s_3 - s_1)^2. \tag{20}$$

This is most easily derived by noticing that, by definition, the right side is $\Delta(G)$, the discriminant of G, and so can be expressed in terms of the coefficients of G using the formula $\Delta(x^3 + px + q) = -4p^3 - 27q^2$ [**5**, 14.18]. Taking square roots of both sides of (20), we get, after a possible reindexing of the zeros of G,

$$2b(27a^2 + b^2) = (s_1 - s_2)(s_2 - s_3)(s_3 - s_1). \tag{21}$$

A straightforward calculation using (19) and (21) now shows that

$$\begin{aligned}&(27a^2 + b^2) f \\ &= (s_1x_1 + s_2x_2 + s_3x_3)(s_2x_1 + s_3x_2 + s_1x_3)(s_3x_1 + s_1x_2 + s_2x_3),\end{aligned} \tag{22}$$

which gives the factorization of f in the case that $27a^2 + b^2 \neq 0$.

If a and b are real numbers, then the factorization can be expressed conveniently using trigonometric functions. After multiplying f by some nonzero number, we can assume that $a = \cos 3\theta$ and $b = 3\sqrt{3}\sin 3\theta$ for some $\theta \in \mathbb{R}$. Then $27a^2 + b^2 = 27$ and

$$G = x^3 - 27x - 54\cos 3\theta = \frac{1}{54}\left(4\left(\frac{x}{6}\right)^3 - 3\left(\frac{x}{6}\right) - \cos 3\theta\right).$$

Because of the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, G has zeros $s_1 = 6\cos\theta$, $s_2 = 6\cos(\theta + 2\pi/3)$ and $s_3 = 6\cos(\theta + 4\pi/3)$. These zeros have been indexed so that (21) holds. (Getting the indexing wrong is equivalent to changing the sign of b.)

After a bit of simplification, the factorization of f in (22) can be written as

$$\begin{aligned}&(\cos 3\theta)(2x_1 - x_2 - x_3)(2x_2 - x_3 - x_1)(2x_3 - x_1 - x_2) \\ &\qquad + (3\sqrt{3}\ \sin 3\theta)(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \\ &= 8(r_1x_1 + r_2x_2 + r_3x_3)(r_2x_1 + r_3x_2 + r_1x_3)(r_3x_1 + r_1x_2 + r_2x_3)\end{aligned} \tag{23}$$

where $r_1 = \cos\theta$, $r_2 = \cos(\theta + 2\pi/3)$, and $r_3 = \cos(\theta + 4\pi/3)$.

For example, with $\theta = 10^\circ$, (23) becomes

$$\begin{aligned}&x_1^3 + x_2^3 + x_3^3 - 3(x_2x_1^2 + x_1x_3^2 + x_3x_2^2) + 6x_1x_2x_3 \\ &= \frac{8}{\sqrt{3}}(r_1x_1 + r_2x_2 + r_3x_3)(r_2x_1 + r_3x_2 + r_1x_3)(r_3x_1 + r_1x_2 + r_2x_3)\end{aligned}$$

where $r_1 = \cos 10^\circ$, $r_2 = \cos 130^\circ$ and $r_3 = \cos 250^\circ$.

Example 4. Suppose that

$$\begin{aligned}f = {}& A(x_1^3 + x_2^3 + x_3^3) + B_1(x_1^2x_2 + x_2^2x_3 + x_1x_3^2) \\ &+ B_2(x_1x_2^2 + x_1^2x_3 + x_2x_3^2) + Cx_1x_2x_3\end{aligned}$$

for some $A, B_1, B_2, C \in \mathbb{C}$. It is not hard to show that such forms are exactly those that are unchanged by the cyclic permutation of the variables $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1$.

To determine the complete reducibility of this form, it is useful to write it as a sum of completely reducible forms

$$\begin{aligned} f &= a(2x_1 - x_2 - x_3)(2x_2 - x_3 - x_1)(2x_3 - x_1 - x_2) \\ &\quad + b(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \\ &\quad + c(x_1 + x_2 + x_3)^3 + d(x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3) \end{aligned}$$

where

$$a = \frac{1}{54}(6A - 3(B_1 + B_2) + 2C) \qquad b = \frac{1}{2}(B_2 - B_1)$$

$$c = \frac{1}{27}(3A + 3(B_1 + B_2) + C) \qquad d = \frac{1}{9}(6A - C).$$

Now we see that Example 3 discusses the special case $c = d = 0$.

The Hessian of f is

$$\mathcal{H} = 162d^2 f - 216S(x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3)$$

where

$$\begin{aligned} S &= (27a^2 + b^2)c + d^3 \\ &= \frac{1}{27}(9A^3 + B_1^3 + B_2^3 + AC^2 - 3AB_1B_2 - 3A^2C - B_1B_2C). \end{aligned}$$

By Theorem 1, f is completely reducible if and only if $\mathcal{H}$ is a multiple of f. This can happen in one of two ways; either f is a multiple of $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$ or $S = 0$. In the first case, f factors as in (16).

How does f factor if $S = 0$? We will assume that $S = 0$ with $c \neq 0$ since otherwise $c = d = 0$ and we are in the situation of Example 3. Let r_1, r_2 and r_3 be the zeros of the cubic polynomial

$$F = 27c(x^3 - x^2) + (9c + 3d)x - (2a + c + d). \tag{24}$$

Matching coefficients in this expression and $F = 27c(x - r_1)(x - r_2)(x - r_3)$ gives

$$\begin{aligned} r_1 + r_2 + r_3 &= 1 \\ 9c(r_1r_2 + r_1r_3 + r_2r_3) &= 3c + d \\ 27c\, r_1r_2r_3 &= 2a + c + d. \end{aligned} \tag{25}$$

Much like in Example 3, these equations, together with $S = 0$, imply

$$2^2b^2 = 27^2c^2\,(r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2.$$

Once again, this equation is most easily derived using the formula for the discriminant of a cubic polynomial [**5**, 14.18]. Taking square roots, we have, after a possible reindexing of the zeros of F,

$$2b = 27c\,(r_1 - r_2)(r_2 - r_3)(r_3 - r_1). \tag{26}$$

Equations (25) and (26) imply that

$$f = 27c(r_1x_1 + r_2x_2 + r_3x_3)(r_2x_1 + r_3x_2 + r_1x_3)(r_3x_1 + r_1x_2 + r_2x_3), \tag{27}$$

and so f is completely reducible.

A nice factorization of this type is obtained by setting $A = 1$, $B_1 = 3$, $B_2 = -4$, and $C = -1$. This corresponds to $a = 7/54$, $b = -7/2$, $c = -1/27$, and $d = 7/9$. Since $S = (27a^2 + b^2)c + d^3 = 0$, f is completely reducible.

The polynomial $F = -x^3 + x^2 + 2x - 1$ in (24) has zeros $r_1 = -2\cos(2\pi/7)$, $r_2 = -2\cos(4\pi/7)$, and $r_3 = -2\cos(6\pi/7)$. This can be easily confirmed by setting $\eta = e^{2\pi i/7}$ so that $r_1 = -(\eta + \eta^6)$, $r_2 = -(\eta^2 + \eta^5)$, and $r_3 = -(\eta^3 + \eta^4)$. Then the expressions obtained by setting $x = r_1, r_2, r_3$ in F reduce to zero because of the equations $\eta^7 = 1$ and $\eta^6 + \eta^5 + \eta^4 + \eta^3 + \eta^2 + \eta + 1 = (\eta^7 - 1)/(\eta - 1) = 0$.

After a bit of simplification, (27) becomes

$$\begin{aligned} &x_1^3 + x_2^3 + x_3^3 + 3(x_1^2x_2 + x_2^2x_3 + x_1x_3^2) - 4(x_1x_2^2 + x_1^2x_3 + x_2x_3^2) - x_1x_2x_3 \\ &\quad = 8(s_1x_1 + s_2x_2 + s_3x_3)(s_2x_1 + s_3x_2 + s_1x_3)(s_3x_1 + s_1x_2 + s_2x_3), \end{aligned}$$

where $s_1 = \cos(2\pi/7)$, $s_2 = \cos(4\pi/7)$, and $s_3 = \cos(6\pi/7)$.

Curiously, because $S = 0$, the substitution $x = (1 - y/d)/3$ in (24) gives

$$F(x) = \frac{1}{27a^2 + b^2}\, G(y),$$

where G is defined in (18)—a polynomial whose coefficients depend on a and b only. Moreover, if $a, b, c, d \in \mathbb{R}$, the same trigonometric trick used in Example 3 can be used to express the factorization of f when it occurs. For example,

$$\begin{aligned} &x_1^3 + x_2^3 + x_3^3 - 3(x_2x_1^2 + x_1x_3^2 + x_3x_2^2) - 3x_1x_2x_3 \\ &\qquad = -\frac{1}{3}(r_1x_1 + r_2x_2 + r_3x_3)(r_2x_1 + r_3x_2 + r_1x_3)(r_3x_1 + r_1x_2 + r_2x_3) \end{aligned}$$

where $r_1 = 1 - 2\cos 20^\circ$, $r_2 = 1 - 2\cos 140^\circ$, and $r_3 = 1 - 2\cos 260^\circ$. The factorization of (4) can be obtained from this equation by setting $x_1 = x$, $x_2 = y$, and $x_3 = 1$.

Example 5. Considerable research has been devoted to the question of the reducibility of $g(x) + h(y)$ where $g(x) \in \mathbb{C}[x]$ and $h(y) \in \mathbb{C}[y]$ are univariate polynomials [**17**, p. 157]. For example, $x^2 - y^2$ is reducible, but $x^3 + y^3 + 1$ is not. With the tools at hand, we are in a position to determine the complete reducibility of $g(x) + h(y)$ in the case that g and h have degree 3.

Suppose that $g(x) = g_0 + g_1x + g_2x^2 + g_3x^3$ and $h(y) = h_0 + h_1y + h_2y^2 + h_3y^3$ with g_3 and h_3 nonzero. To apply Theorem 1, we replace x and y by x_1 and x_2 and then homogenize using x_3. Thus, $g(x) + h(y)$ is completely reducible if and only if

$$f = (g_0 + h_0)\, x_3^3 + (g_1x_1 + h_1x_2)\, x_3^2 + (g_2x_1^2 + h_2x_2^2)\, x_3 + g_3\, x_1^3 + h_3\, x_2^3$$

is completely reducible. The Hessian of f is

$$\begin{aligned}\mathcal{H} = &-24h_3(g_2^2 - 3g_1g_3)x_1^2x_2 - 8h_2(g_2^2 - 3g_1g_3)x_1^2x_3 \\ &- 24g_3(h_2^2 - 3h_1h_3)x_1x_2^2 - 8g_2(h_2^2 - 3h_1h_3)x_2^2x_3 \\ &- 24(g_3h_1h_2 + g_1g_2h_3 - 9g_0g_3h_3 - 9g_3h_0h_3)x_1x_2x_3 \\ &- 8(3g_3h_1^2 + g_1g_2h_2 - 9g_0g_3h_2 - 9g_3h_0h_2)x_1x_3^2 \\ &- 8(g_2h_1h_2 + 3g_1^2h_3 - 9g_0g_2h_3 - 9g_2h_0h_3)x_2x_3^2 \\ &- 8(g_2h_1^2 + g_1^2h_2 - 3g_0g_2h_2 - 3g_2h_0h_2)x_3^3.\end{aligned}$$

Suppose that f is completely reducible. Then, by Theorem 1, $\mathcal{H}$ is a multiple of f. Since the coefficients of $x_1^2x_2$, $x_1x_2^2$, and $x_1x_2x_3$ in f are zero, the coefficients of $x_1^2x_2$, $x_1x_2^2$, and $x_1x_2x_3$ in $\mathcal{H}$ must also be zero. Hence,

$$g_2^2 - 3g_1g_3 = h_2^2 - 3h_1h_3 = g_3h_1h_2 + g_1g_2h_3 - 9g_0g_3h_3 - 9g_3h_0h_3 = 0. \qquad (28)$$

What do these conditions say about f? A straightforward calculation (without any assumption about reducibility) gives

$$\begin{aligned}27g_3^2h_3^2\, f = &\; g_3^2(3h_3x_2 + h_2x_3)^3 + h_3^2(3g_3x_1 + g_2x_3)^3 \\ &- 3g_3h_3(g_3h_1h_2 + g_1g_2h_3 - 9g_0g_3h_3 - 9g_3h_0h_3)x_3^3 \\ &- h_3^2(g_2^2 - 3g_1g_3)(9g_3x_1 + g_2x_3)\, x_3^2 \\ &- g_3^2(h_2^2 - 3h_1h_3)(9h_3x_2 + h_2x_3)\, x_3^2.\end{aligned}$$

So, if f is completely reducible, then, using (28), f can be written as

$$f = \frac{1}{27h_3^2}(3h_3x_2 + h_2x_3)^3 + \frac{1}{27g_3^2}(3g_3x_1 + g_2x_3)^3,$$

that is, f is a sum of two cubes.

The converse is also true. If f is a sum of the cubes of two linear forms as above, then f is completely reducible by the argument in Example 2.

Returning to the original question, suppose that $g(x) \in \mathbb{C}[x]$ and $h(y) \in \mathbb{C}[y]$ are cubic univariate polynomials. Then $g(x) + h(y)$ is completely reducible if and only if $g(x) + h(y) = (ax + b)^3 + (cy + d)^3$ for suitable $a, b, c, d \in \mathbb{C}$.

5. PROOF OF THEOREM 1. In this section, we complete the proof of Theorem 1 by showing that, if f is a ternary quadratic form such that $\mathcal{H}(f) = 0$ or if f is a ternary cubic form such that $\mathcal{H}(f)$ is a multiple of f, then f is completely reducible. This part of the proof requires understanding the small bit of invariant theory that applies to Hessians. Since the properties we need hold for ternary forms of arbitrary degree, we will suppose that $f = f(x_1, x_2, x_3)$ is a ternary form of arbitrary degree, and we consider how f and its Hessian are affected by a linear change of variables of the form

$$\begin{aligned}x_1 &= u_1\, y_1 + v_1\, y_2 + w_1\, y_3 \\ x_2 &= u_2\, y_1 + v_2\, y_2 + w_2\, y_3 \\ x_3 &= u_3\, y_1 + v_3\, y_2 + w_3\, y_3\end{aligned} \qquad (29)$$

where (y_1, y_2, y_3) is a vector of variables and $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ are in $\mathbb{C}^3$. We assume that (29) is an invertible change of variables, that is, the transformation matrix

$$A = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

is invertible. This holds if and only $\det A \neq 0$, if and only if $\{u, v, w\}$ is linearly independent.

Now define

$$F = f(u_1 y_1 + v_1 y_2 + w_1 y_3,\ u_2 y_1 + v_2 y_2 + w_2 y_3,\ u_3 y_1 + v_3 y_2 + w_3 y_3). \qquad (30)$$

It is not hard to see that $F = F(y_1, y_2, y_3)$ is a ternary form in the new variables (y_1, y_2, y_3) with the same degree as f. Other properties of f that are important to us are also preserved by this change of variables. For example, if f is a product of two forms, then so is F, preserving the degrees of the factors. In particular, F is completely reducible if and only if f is completely reducible.

The Hessian of F and the Hessian of f are related by

$$\mathcal{H}(F) = (\det A)^2\, \mathcal{H}(f). \qquad (31)$$

It has to be understood that both sides of this equation are functions of (y_1, y_2, y_3). To evaluate $\mathcal{H}(f)$ at a particular point (y_1, y_2, y_3), the first step is to find (x_1, x_2, x_3) from (29). Then these values are plugged into the expression for $\mathcal{H}(f)$ as function of (x_1, x_2, x_3).

If the degree of f is small, such as two or three, (31) can be confirmed by direct calculation. Deriving this equation in the general case requires expressing the partial derivatives of F with respect to the ys in terms of the partial derivatives of f with respect to the xs. For example, by the chain rule,

$$\partial_1 F = \frac{\partial F}{\partial y_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_1} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial y_1} = u_1 \partial_1 f + u_2 \partial_2 f + u_3 \partial_3 f.$$

This and similar equations for $\partial_2 F$ and $\partial_3 F$ can be written compactly in matrix notation as $\nabla F = A^T \nabla f$. Taking a further partial derivative of this equation, we get

$$\begin{aligned}\partial_{12}^2 F = \frac{\partial F}{\partial y_1 \partial y_2} &= u_1 v_1 \partial_{11}^2 f + u_1 v_2 \partial_{12}^2 f + u_1 v_3 \partial_{13}^2 f + u_2 v_1 \partial_{12}^2 f + u_2 v_2 \partial_{22}^2 f \\ &\quad + u_2 v_3 \partial_{23}^2 f + u_3 v_1 \partial_{13}^2 f + u_3 v_2 \partial_{23}^2 f + u_3 v_3 \partial_{33}^2 f.\end{aligned}$$

Once again, this and similar equations are better written in matrix notation:

$$\begin{bmatrix} \partial_{11}^2 F & \partial_{12}^2 F & \partial_{13}^2 F \\ \partial_{12}^2 F & \partial_{22}^2 F & \partial_{23}^2 F \\ \partial_{13}^2 F & \partial_{23}^2 F & \partial_{33}^2 F \end{bmatrix} = A^T \begin{bmatrix} \partial_{11}^2 f & \partial_{12}^2 f & \partial_{13}^2 f \\ \partial_{12}^2 f & \partial_{22}^2 f & \partial_{23}^2 f \\ \partial_{13}^2 f & \partial_{23}^2 f & \partial_{33}^2 f \end{bmatrix} A.$$

Taking the determinant of both sides of this equation and using well-known properties of determinants gives (31).

For the upcoming discussion of quadratic forms, we note that (31) implies that $\mathcal{H}(f) = 0$ if and only if $\mathcal{H}(F) = 0$.

With a view to the condition on cubic forms in Theorem 1, suppose that $\mathcal{H}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$. Evaluating this with (x_1, x_2, x_3) given by (29), we get on the left $\mathcal{H}(f)$ with the meaning it has in (31) and on the right $\lambda F(y_1, y_2, y_3)$. Thus,

$$\mathcal{H}(F) = \lambda(\det A)^2 F. \tag{32}$$

The important point is that, if $\mathcal{H}(f)$ is a multiple of f, then $\mathcal{H}(F)$ is a multiple of F. The converse is also true since A is invertible.

We need one other small result that holds for ternary forms of arbitrary degree.

Lemma 1. *Suppose that f is a ternary form. Then there are $u, v \in \mathbb{C}^3$ such that $f(u) = f(v) = 0$ and $\{u, v\}$ is linearly independent.*

Proof. We carry out the proof only for the cubic case. Only notation changes are needed to make the proof valid for forms of arbitrary degree.

Let f be a cubic form as in (10). Suppose first that $f_{111} \neq 0$. Then $f(x, 1, 0) = f_{111}x^3 + f_{112}x^2 + f_{122}x + f_{222} \in \mathbb{C}[x]$ is a cubic univariate polynomial in x so has a zero. In other words, there is some $r \in \mathbb{C}$ such that $f(r, 1, 0) = 0$. Similarly, there is some $s \in \mathbb{C}$ such that $f(s, 0, 1) = 0$. The claim is now true with $u = (r, 1, 0)$ and $v = (s, 0, 1)$. And, by symmetry, the claim is also true if $f_{222} \neq 0$ or $f_{333} \neq 0$.

It remains only to consider the case that $f_{111} = f_{222} = f_{333} = 0$. When this happens, we have $f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1) = 0$, and so the claim is true with u and v being any two of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. ■

We now show that the main theorem holds for ternary quadratic forms in a special case.

Lemma 2. *Suppose that f is a ternary quadratic form as in* (8) *with $f_{11} = f_{22} = 0$. If $\mathcal{H}(f) = 0$, then f is completely reducible.*

Proof. Setting $f_{11} = f_{22} = 0$ in (9), we find $\mathcal{H} = 2f_{12}(f_{13}f_{23} - f_{12}f_{33})$. Since we are assuming $\mathcal{H} = 0$, the proof splits into two cases.

1. If $f_{12} = 0$, then $f = x_3(f_{13}x_1 + f_{23}x_2 + f_{33}x_3)$ and f is completely reducible.
2. If $f_{12} \neq 0$, then we have $f_{13}f_{23} - f_{12}f_{33} = 0$ and the claim follows from the identity

$$f_{12}f = (f_{12}x_2 + f_{13}x_3)(f_{12}x_1 + f_{23}x_3) - (f_{13}f_{23} - f_{12}f_{33})x_3^2. \quad ■$$

The following theorem completes the proof of the main theorem for ternary quadratic forms.

Theorem 2. *Let f be a ternary quadratic form. If $\mathcal{H}(f) = 0$, then f is completely reducible.*

Proof. By Lemma 1, there are $u, v \in \mathbb{C}^3$ such that $f(u) = f(v) = 0$ and $\{u, v\}$ is linearly independent. Choose a third vector $w \in \mathbb{C}^3$ such that $\{u, v, w\}$ is independent.

Define $F(y_1, y_2, y_3)$ by (30), written in expanded form as

$$F(y_1, y_2, y_3) = F_{11}y_1^2 + F_{22}y_2^2 + F_{33}y_3^2 + F_{12}y_1y_2 + F_{13}y_1y_3 + F_{23}y_2y_3.$$

Then, by construction, $F_{11} = F(1, 0, 0) = f(u) = 0$ and $F_{22} = F(0, 1, 0) = f(v) = 0$. By (31), $\mathcal{H}(F) = 0$ and so Lemma 2 implies that F is completely reducible. This in turn implies that f is completely reducible. ■

The proof of the main theorem in the cubic case follows the same path we have just followed for quadratic forms: First we prove the claim in a special case, then we reduce the general case to the special one.

Let f be a ternary cubic form as in (10) with $f_{111} = f_{222} = 0$. Set

$$\Lambda = 2(f_{123}^2 - 4f_{122}f_{133} + 8f_{113}f_{223} - 4f_{112}f_{233})$$

and $g = \mathcal{H} - \Lambda f$. Then g is a ternary cubic form whose coefficients we label as g_{ijk} following the pattern in (10). We have chosen Λ so that $g_{112} = g_{122} = 0$. As a consequence, if $\mathcal{H} = \lambda f$ for some $\lambda \in \mathbb{C}$ and either f_{112} or f_{122} is nonzero, then $\lambda = \Lambda$ and hence $g = 0$.

Other coefficients that are needed for our discussion are

$$\begin{aligned}
g_{111} &= 8\left(f_{112}f_{113}f_{123} - f_{113}^2f_{122} - f_{112}^2f_{133}\right)\\
g_{222} &= 8\left(f_{122}f_{223}f_{123} - f_{223}^2f_{112} - f_{122}^2f_{233}\right)\\
g_{223} &= 24\left(f_{122}f_{133}f_{223} - f_{223}^2f_{113} - f_{122}^2f_{333}\right)\\
g_{123} &= 24\left(f_{112}f_{133}f_{223} + f_{113}f_{122}f_{233} - f_{113}f_{123}f_{223} - f_{112}f_{122}f_{333}\right).
\end{aligned} \tag{33}$$

Lemma 3. *Suppose that f is a ternary cubic form as in* (10) *with $f_{111} = f_{222} = 0$. If $\mathcal{H}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, then f is completely reducible.*

Proof. The proof splits into cases depending on f_{112} and f_{122}.

1. Suppose that f_{112} and f_{122} are both nonzero. As explained above, this implies that $g = 0$. It turns out that to show that f is completely reducible, it suffices that g_{111}, g_{222} and g_{123} are zero. This is because (without any assumptions except $f_{111} = f_{222} = 0$),

$$f_{112}^2f_{122}^2f = abc - \frac{1}{24}Lx_3^2 \tag{34}$$

where a, b, c and L are linear forms defined by

$$\begin{aligned}
a &= f_{112}x_2 + f_{113}x_3\\
b &= f_{122}x_1 + f_{223}x_3\\
c &= f_{112}^2f_{122}x_1 + f_{112}f_{122}^2x_2\\
&\quad + (f_{112}f_{122}f_{123} - f_{113}f_{122}^2 - f_{112}^2f_{223})x_3\\
L &= 3f_{122}^2g_{111}\,x_1 + 3f_{112}^2g_{222}\,x_2\\
&\quad + (f_{112}f_{122}g_{123} + 3f_{122}f_{223}g_{111} + 3f_{112}f_{113}g_{222})\,x_3.
\end{aligned}$$

Since g_{111}, g_{222} and g_{123} are zero, $L = 0$ and so, by (34), f is a nonzero multiple of abc, showing that f is completely reducible.

2. Suppose that exactly one of f_{112} and f_{122} is zero. Without loss of generality, we can assume that $f_{112} = 0$ and $f_{122} \neq 0$. As above, the assumption that $\mathcal{H}$ is a multiple of f implies that $g = 0$. Setting $f_{112} = 0$ in (33), we get $g_{111} = -8 f_{113}^2 f_{122}$ and so the equations $g_{111} = 0$ and $f_{122} \neq 0$ imply that $f_{113} = 0$. This leaves only six potentially nonzero coefficients of f. With no assumptions except $f_{111} = f_{222} = f_{112} = f_{113} = 0$, a straightforward calculation gives

$$f_{122}^2 f = f_{122}(f_{122}x_1 + f_{223}x_3)(f_{122}x_2^2 + f_{123}x_2x_3 + f_{133}x_3^2) - \frac{1}{24}(3g_{222}x_2 + g_{223}x_3)x_3^2.$$

Since, in addition, $g_{222} = g_{223} = 0$, this equation can be written as

$$f = \frac{1}{f_{122}}(f_{122}x_1 + f_{223}x_3)(f_{122}x_2^2 + f_{123}x_2x_3 + f_{133}x_3^2).$$

The quadratic factor of f is reducible because it is a binary form in the variables x_2 and x_3, and so f is completely reducible.

3. Suppose that $f_{112} = f_{122} = 0$. Then

$$f = x_3\left(f_{113}x_1^2 + f_{123}x_1x_2 + f_{223}x_2^2 + f_{133}x_1x_3 + f_{233}x_2x_3 + f_{333}x_3^2\right). \quad (35)$$

An easy calculation shows that

$$\mathcal{H} = 2(f_{123}^2 - 4f_{113}f_{223})f + 4\,\mathcal{H}_2\,x_3^3$$

with

$$\mathcal{H}_2 = 8f_{113}f_{223}f_{333} + 2f_{123}f_{133}f_{233} - 2f_{133}^2f_{223} - 2f_{113}f_{233}^2 - 2f_{123}^2f_{333}.$$

Note that $\mathcal{H}_2$ is the Hessian of the quadratic factor of f in (35). Since $\mathcal{H}$ is a multiple of f, we have $\mathcal{H}_2\,x_3^3$ is a multiple of f. This implies that either f is a multiple of x_3^3 or $\mathcal{H}_2 = 0$. In the first case, f is obviously completely reducible. If $\mathcal{H}_2 = 0$, then, by Theorem 2, the quadratic factor of f in (35) is reducible, and f is completely reducible. ■

Our final task is to show that if f is an arbitrary ternary cubic form such that $\mathcal{H}(f)$ is a multiple of f, then f is completely reducible. This proof is almost a word-by-word translation of Theorem 2 to cubic forms.

Theorem 3. *Let f be a ternary cubic form. If $\mathcal{H}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, then f is completely reducible.*

Proof. By Lemma 1, there are $u, v \in \mathbb{C}^3$ such that $f(u) = f(v) = 0$ and $\{u, v\}$ is linearly independent. Choose a third vector $w \in \mathbb{C}^3$ such that $\{u, v, w\}$ is independent. Define $F(y_1, y_2, y_3)$ by (30), written in expanded form as

$$F = F_{111}y_1^3 + F_{112}y_1^2y_2 + F_{122}y_1y_2^2 + F_{222}y_2^3 + F_{113}y_1^2y_3 + F_{123}y_1y_2y_3 + F_{223}y_2^2y_3 + F_{133}y_1y_3^2 + F_{233}y_2y_3^2 + F_{333}y_3^3.$$

Then, by construction, $F_{111} = F(1, 0, 0) = f(u) = 0$ and $F_{222} = F(0, 1, 0) = f(v) = 0$. By (32), $\mathcal{H}(F)$ is a multiple of F and so Lemma 3 implies that F is completely reducible. This in turn implies that f is completely reducible. ■

6. FURTHER READING. Here are a few questions about the reducibility of forms that we have left unanswered.

1. When is a ternary cubic form f reducible but not completely reducible? That is, when is $f = ab$ where a and b are forms of degrees 1 and 2 with b irreducible? This turns out to be a much more complicated question than the one discussed in this article. Answers are given in [**4**], [**15**, p. 338], and [**17**, p. 213].
2. What about forms with more variables? If f is a quadratic form in $n \geq 2$ variables, then f is reducible if and only if the rank of the Hessian matrix of f is 2 or less. Here, the Hessian matrix is the matrix of second partial derivatives of f whose determinant is the Hessian of f. See [**13**] for a proof and the reason that this result is rather more obscure that it should be.
3. What about forms with higher degree? A condition for the complete reducibility of a ternary form of arbitrary degree is given in [**3**]. This result is generalized further in [**6**, Theorem 2.12, p. 144].

REFERENCES

1. S. Aronhold, Zur Theorie der homogenen Functionen dritten Grades von drei Variablen, *J. Reine Angew. Math.* **39** (1849) 140–159.
2. A. Brill, Über symmetrische Functionen von Variabelnparren, *Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen* **20** (1893) 757–762.
3. ———, Über die Zerfällung einer Ternärform in Linearfactoren, *Math. Ann.* **50** (1898) 157–182.
4. L. Copeland, Matrix conditions for multiple points of a ternary cubic, *Ann. of Math.* **31** no. 2 (1930) 629–632.
5. D. Dummit, R. Foote, *Abstract Algebra*. Third edition. Wiley, Hoboken, 2004.
6. I. Gelfand, M. Kapranov, A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Basel, 1994.
7. P. Gordan, Das Zerfallen der Curven in gerade Linien, *Math. Ann.* **45** (1894) 410–427.
8. J. Grace, A. Young, *The Algebra of Invariants*. Cambridge Univ. Press, Cambridge, 1903.
9. S. Gundelfinger, Zur Theorie der ternären cubischen Formen, *Math. Ann.* **4** (1871) 144–163.
10. O. Hesse, Über die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variablen, *J. Reine Angew. Math.* **28** (1844) 68–96.
11. ———, Über die Wendepuncte der Curven dritter Ordnung, *J. Reine Angew. Math.* **28** (1844) 97–107.
12. E. Kaltofen, Polynomial factorization 1987–1991, in *LATIN '92*, Lecture Notes in Comput. Sci., Vol. 583, Springer, Berlin, 1992. 294–313.
13. B. Kronenthal, F. Lazebnick, When can you factor a quadratic form?, *Math. Mag.* **87** (2014) 25–36.
14. P. Olver, *Classical Invariant Theory*. Cambridge Univ. Press, Cambridge, 1999.
15. E. Pascal, *Repertorium der Höheren Mathematik I: Analysis*. B. G. Teubner, Leipzig, 1900.
16. G. Salmon, *A Treatise on the Higher Plane Curves*. Hodges and Smith, Dublin, 1852.
17. A. Schinzel, *Polynomials with Special Regard to Reducibility*. Cambridge Univ. Press, Cambridge, 2000.
18. A. Thaer, Über die Zerlegbarkeit einer ebenen Linie dritter Ordnung in drei gerade Linien, *Math. Ann.* **14** (1879) 545–556.

GARY BROOKFIELD received his Ph.D. from the University of California, Santa Barbara in 1997. After visiting positions at the University of Wisconsin, the University of Iowa, and the University of California, Riverside, he has been at California State University, Los Angeles since 2003.
Department of Mathematics, California State University, Los Angeles, CA 90032-8204
gbrookf@calstatela.edu