Circular Coloring for Graphs with Distance Constraints *

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Abstract

Let $G = (V, E)$ be a simple un-weighted graph, and let $\vec{d} = (d_1, d_2, \cdots, d_m)$ be a sequence of positive reals. For a positive real r, let S_r denote the circle on R^2 centered at the origin with circumference r. A circular $r-$coloring for G with distance constraint $\vec{d}$ is a mapping $f : V(G) \to S_r$ such that $|f(u) - f(v)|_r \geq d_i$, whenever the distance between u and v in G is i (where $|x - y|_r$ is the length of a shorter arc between x and y on S_r). The circular chromatic number of G with distance constraint $\vec{d}$, denoted by $\chi_c(G(\vec{d}))$, is the infimum of r such that there exists a circular $r-$coloring for G with distance constraints $\vec{d}$. For any cycle C_n, $n \geq 3$, we determine the value of $\chi(C_n(d, 1))$, expressed as a continuous, piecewise linear function of d, $d > 0$. In addition, we discuss relations between circular coloring (for weighted graphs) and integral distance labeling.

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1 Introduction

Introduced by Mohar [9], circular coloring for edge weighted graphs is a generalization of conventional circular coloring for simple graphs. An *edge weighted graph* with vertex set V is a pair $G = (V, A)$, where $A : V \times V \to R^+ \cup \{0\}$ is a weight assignment. For each $(u, v) \in V \times V$, we write $a_{uv} = A(u, v)$. For a positive real r, denote $S_r \subset R^2$ the circle with circumference r centered at the origin of R^2. For any $x, y \in S_r$, let $l(x, y)$ denote the arc length from x to y, in the clockwise direction. A *circular* $r-$*coloring* of $G = (V, A)$ is a function, $c : V \to S_r$, such that $l(c(u), c(v)) \geq a_{uv}$ for every $(u, v) \in V \times V$. The *circular chromatic number* $\chi_c(G)$ of an edge-weighted graph $G = (V, A)$ is the infimum of all real numbers r for which there exists a circular $r-$coloring of G.

The weights are *weakly symmetric* if the following is satisfied: For any $u, v \in V$, if $a_{uv} = 0$, then $a_{vu} = 0$. The weights are called *symmetric* if $a_{uv} = a_{vu}$, for every $u, v \in V$. Mohar [9] proved that the infimum in the above definition of $\chi_c(G)$ for edge weighted graphs can be replaced by minimum, if the weights are weakly symmetric.

We investigate circular coloring for weighted graphs with distance constraints. For any $u, v \in V$, let $\text{dist}_G(u, v)$ denote the *distance* (length of a shortest path) between u and v in G; when G is clear in the context, we simply denote $\text{dist}_G(u, v)$ by $\text{dist}(u, v)$. Let $\vec{d} = (d_1, d_2, \cdots, d_m)$ be a sequence of positive reals. The *graph* G *with distance constraint* $\vec{d}$, denoted by $G(d_1, d_2, \cdots, d_m) = G(\vec{d}) = (V, A)$, is a symmetric edge weighted graph, defined as follows. For each $u, v \in V(G)$, let $a_{uv} = a_{vu} = d_i$, if $\text{dist}_G(u, v) = i$ where $i = 1, 2, \cdots, m$; otherwise $a_{uv} = 0$. Following this notion, the conventional circular chromatic number $\chi_c(G)$ of a simple un-weighted graph $G = (V, E)$ is the case when $\vec{d} = (1)$, that is, $\chi_c(G) = \chi_c(G(1))$; and the conventional circular chromatic number of G^2, the square of G (by adding edges between vertices of distance two apart), has $\chi_c(G^2) = \chi_c(G(1, 1))$.

Section 3 of this article is devoted to complete solutions of $\chi_c(C_n(d, 1))$, for any cycle C_n and any $d > 0$. For any $n \geq 3$, we give the formula of $\chi_c(C_n(d,1))$, expressed as a continuous, piecewise linear function of d.

Let $\vec{d} = (d_1, d_2, \cdots, d_m)$ be positive reals. For a simple un-weighted graph $G = (V, E)$, circular coloring for $G(\vec{d})$ is closely related to integral circular distance labeling. For a positive integer k and a graph $G = (V, E)$, a *circular* $(k; \vec{d})-$*labeling* of G is a function, $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$, such that the following is satisfied:

$$|f(x) - f(y)|_k \geq d_i, \text{if } \mathrm{dist}_G(x, y) = i \text{ and } i = 1, 2, \cdots, m,$$

where $|x - y|_k = \min\{|x - y|, k - |x - y|\}$. The $\sigma(d_1, d_2, \cdots, d_m)$-*number* (or $\sigma_{\vec{d}}$-*number*) of an un-weighted simple graph G, denoted by $\sigma_{\vec{d}}(G)$, is the smallest integer k such that G admits a circular $(k; \vec{d})-$labeling. The special case when $\vec{d} = (d_1, d_2)$ is also known as the *circular distance two labeling*; the values of $\sigma_{d_1,d_2}(G)$ for some families of graphs have been studied in [4, 5, 6, 7, 8].

In the next section, we establish the following relation for any simple graph G and any $\vec{d} = (d_1, d_2, \cdots, d_m)$ of positive integers d_i:

$$\sigma_{\vec{d}}(G) - 1 < \chi_c(G(\vec{d})) \leq \sigma_{\vec{d}}(G). \tag{1.1}$$

The upper bound in (1.1) is sharp for some graphs. The result on the values of $\chi_c(C_n(d,1))$ for cycles obtained in this article implies that the lower bound in (1.1) is sharp, in the sense that there exist graphs G such that the values of $\chi_c(G(\vec{d}))$ approach to the lower bound, as closely as possible.

2 Basic Properties

For any reals a, b with $a \leq b$, we denote the half-open interval $[a, b)$ by the set of all reals x, $a \leq x < b$. For any real r, we regard S_r as $[0, r)$, by fixing any point on S_r as 0, and going in the clockwise direction. Thus, a circular $r-$coloring for a weighted graph can be viewed as a mapping from the vertex set to $[0, r)$.

Theorem 1 *Let $G = (V, A)$ be a finite symmetric weighted graph with rational weights. If there is a circular $r-$coloring for $G = (V, A)$ for some rational r, then there exists a circular $r-$coloring f for G such that $f(u)$ is rational for every $u \in V(G)$.*

Proof. Let g be a circular $r-$coloring for $G = (V, A)$, where r is rational. Let q be a common denumerator of r and all the weights (expressed as fractions). Let $f(u) = \lfloor qg(u) \rfloor / q$, for every $u \in V(G)$. It is straightforward to verify that f is a circular $r-$coloring for $G = (V, A)$. ∎

For any real t and any $\vec{d} = (d_1, d_2, \cdots, d_m)$, let $t\vec{d} = (td_1,\ td_2,\ \cdots,\ td_m)$. The following is obvious.

Lemma 2 *Let G be a simple un-weighted graph, and let $\vec{d} = (d_1, d_2, \cdots, d_m)$ be positive reals. Then $t\ \chi_c(G(\vec{d})) = \chi_c(G(t\vec{d}))$ holds for any real t.*

By Lemma 2, finding the value of $\chi_c(G(d_1, d_2))$ for any positive reals d_1 and d_2, bounds to determining $\chi_c(G(d, 1))$ for any positive real d.

Theorem 3 *Let G be a simple un-weighted graph, and let $\vec{d} = (d_1, d_2, \cdots, d_m)$ be positive integers. For any positive integer q,*

$$\chi_c(G(\vec{d})) = \min\{p/q : \textit{there exists a circular } (p; q\vec{d})\textit{−labeling for } G\}.$$

Proof. Since $G(\vec{d})$ is symmetric with integral weights, so $\chi_c(G(\vec{d}))$ is rational (cf. [9]). Let $\chi_c(G(\vec{d})) = p/q$. By the proof of Theorem 1, there exists a circular $(p/q)-$coloring for $G(\vec{d})$ such that for every vertex u, $f(u) = x/q$ for some $x \in \{0, 1, 2, \cdots, p-1\}$. Define $f^*(u) = qf(u)$, $u \in V(G)$. Then f^* is a circular $(p; q\vec{d})-$labeling for G.

On the other hand, for any circular $(p; q\vec{d})-$labeling f of G, the function g, defined by $g(u) = f(u)/q$, $u \in V(G)$, is a circular $(p/q)-$coloring for $G(\vec{d})$. ∎

We now prove (1.1).

Theorem 4 *For any simple graph G, and any $\vec{d} = (d_1, d_2, \cdots, d_m)$ of positive integers, $\sigma_{\vec{d}}(G) = \lceil \chi_c(G(\vec{d})) \rceil$.*

Proof. If f is a circular $(k; \vec{d})-$labeling for G for some integer k, then f is also a circular $k-$coloring for $G(\vec{d})$. Hence $\sigma_{\vec{d}}(G) \geq \lceil \chi_c(G(\vec{d})) \rceil$, as $\sigma_c(G)$ is an integer.

Let f be a circular $r-$coloring for $G(\vec{d})$. Then f generates a circular $(\lceil r \rceil; \vec{d})-$labeling f' for G, defined as $f'(v) = \lfloor f(v) \rfloor$ for every $v \in V(G)$. If $\text{dist}_G(u,v) = i = 1, 2, \cdots, m$, then $|f(u) - f(v)|_p \geq d_i$, implying $|f'(u) - f'(v)|_p \geq d_i$. ∎

The *diameter* of an un-weighted connected graph G, denoted by $\text{diam}(G)$, is the maximum distance over all pairs of vertices in G. For any graph $G = (V, E)$, let G^c denote the *complement* of G. The *path covering number* (or *linear arboricity*) of a graph G, denoted by $p_v(G)$, is the smallest number of paths partitioning $V(G)$. The following was proved in [5].

Theorem 5 *[5] Let G be an n-vertex graph. Then*

$$\sigma_{2,1}(G) \begin{cases} \leq n, & \text{if } G^c \text{ is Hamiltonian;} \\ = n + p_v(G^c), & \text{if } G^c \text{ is not Hamiltonian.} \end{cases}$$

A special case of Theorem 5 is when the diameter of G is two, for which by Theorem 5, and by a discussion in [9] ((c) in Section 1), we have:

Corollary 6 *Let G be an n-vertex graph with diameter two. Then*

$$\sigma_{2,1}(G) = \chi_c(G(2,1)) = \begin{cases} n, & \text{if } G^c \text{ is Hamiltonian;} \\ n + p_v(G^c), & \text{if } G^c \text{ is not Hamiltonian.} \end{cases}$$

A special case to Corollary 6 is when G contains a universal vertex, in which $\sigma_{2,1}(G) = \chi_c(G(2,1)) = n + p_v(G^c)$.

Let $G = (V, A)$ be a weighted graph. A *subgraph* $H = (V', A')$ of G is a weighted graph, with $V' \subset V$, and $A'(u,v) \leq A(u,v)$ for any $u, v \in V'$. If $H(V', A')$ is a subgraph of $G = (V, A)$, then a circular $p-$coloring of $G = (V, A)$, when restricted to V', is a circular $p-$coloring for $H = (V', A')$. Hence, we have

Proposition 7 *Let $G = (V, A)$ be a weighted graph, and $H = (V', A')$ a subgraph of G. Then $\chi_c(H) \leq \chi_c(G)$.*

The following is obtained directly by a *greedy (first-fit)* algorithm.

Proposition 8 *Let T be a tree with maximum degree Δ. Then $\chi_c(T(d, 1)) = 2d + \Delta - 1$, for any positive real d.*

Proposition 9 *Let G be a graph with maximum degree Δ. Then $\chi_c(G(d, 1)) \geq 2d + \Delta - 1$, for any positive real d.*

3 Cycles

For any cycle C_n, we denote the vertex set by $V(C_n) = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, where $v_i \sim v_{i+1}$ for any i, and the sub-index is taken under modular n, for instance, $v_n = v_0$. For any two positive real numbers x and y, if $x = qy + r$ for some integer q and real r, $0 \leq r < y$, then we write $r = x \pmod{y}$. Assume f is a circular $p-$coloring for $C_n(d, 1)$. For any two points, x and y, on $S_p = [0, p)$, we denote $[x, y]$ as the closed arc (interval) from x to y, in the clockwise direction. Similarly, $[a, b)$ denotes a half-open arc (interval) on S_p.

Note, by Prop. 9, $\chi_c(C_n(d, 1)) \geq 2d + 1$, for any n and d.

Theorem 10 *If $0 < d \leq 1/2$, then*

$$\chi_c(C_n(d,1)) = \begin{cases} 2, & \text{if } n \equiv 0 \pmod{4}; \\ 2 + 1/k, & \text{if } n = 4k + 2 \text{ or } n = 2k + 1 \text{ for some } k \geq 1. \end{cases}$$

Proof. Assume n is even, $n = 2m$ for some $m \geq 2$. Then the subgraph induced by edges of weight 1 form two disjoint weighted m-cycles $C_m(1)$, A and B, where $V(A) = \{v_{2i} : i = 0, 1, 2, \cdots, m - 1\}$ and $V(B) = \{v_{2i+1} : i = 0, 1, \cdots, m - 1\}$, as subgraphs in $C_n(d, 1)$. By Prop. 7, $\chi_c(C_n(d, 1)) \geq \chi_c(C_m(1))$. It is known [10] that $\chi_c(C_m(1)) = 2$, if m is even; and $\chi_c(C_m(1)) = 2 + 1/m'$, if $m = 2m' + 1$. Therefore, the lower bounds for even cycles are obtained. To prove the upper bounds, we let $\chi_c(C_m(1)) = p$, and let f be a circular $p-$coloring for A. We then extend f to $C_n(d,1)$ by letting $f(v_{2i+1}) = f(v_i) + 1/2 \pmod p$. It is easy to check that f is a circular $p-$coloring for $C_n(d,1)$, as $d \le 1/2$. So, the upper bounds hold.

Assume $n = 2k+1$. Then all the edges of weight 1 in $C_n(d,1)$ form an n-cycle $C_n(1)$ as a subgraph. By Prop. 7, we have $\chi_c(C_n(d,1)) \ge \chi_c(C_n(1)) = 2 + 1/k$. Moreover, since $d \le 1/2$, it is easy to see that a circular $p-$coloring for $C_n(1)$, with $p = 2+1/k$, is also a circular $p-$coloring for $C_n(d,1)$. ∎

Theorem 11 *Let C_n be a cycle and let $d > 1/2$ be real. If $n = 2k+1$ and $d \ge k$, then $\chi_c(C_n(d,1)) = d(2+1/k) = \frac{nd}{k}$.*

Proof. Assume $n = 2k+1$ and $k \le d$. Let $p = d(2+1/k)$. The function f defined on $V(C_n)$ by $f(v_i) = id \pmod p$ is a circular $p-$coloring for $C_n(d,1)$. Hence, $\chi_c(C_n(d,1)) \le d(2+1/k)$. By Lemma 2, $\chi_c(C_n(d)) = d\chi_c(C_n(1)) = d(2+1/k)$. Hence $\chi_c(C_n(d,1)) \ge \chi_c(C_n(d)) = d(2+1/k)$. ∎

Lemma 12 *The in-equality $\chi_c(C_n(d,1)) \le 2d+2$ holds for: 1) n is even and $d > 1/2$; 2) $n = 2k+1 \ge 9$ and $1/2 < d < k$; and 3) $n = 7$ and $1 \le d \le 2$.*

Proof. It suffices to find a circular $(2d+2)-$coloring for $C_n(d,1)$, for each case. We express such a coloring f by a *difference sequence* $(t_1, t_2, \cdots, t_n)$ of positive reals t_i, where $f(v_0) = 0$ and $f(v_{i+1}) = f(v_i) + t_i \pmod{2d+2}$. The following claim follows from the definition.

Claim. Let $f : V(C_n) \to [0, 2d+2)$ be a function with $f(v_0) = 0$, and $f(v_{i+1}) = f(v_i) + t_i \pmod{2d+2}$, $0 \le i \le n-1$. Then f is a circular $(2d+2)-$coloring for $C_n(d,1)$ if the following hold for all i:

(a) $d \le t_i \le d+2$,

(b) $t_i + t_{i+1} \in [2d, 2d+1] \cup [1,2]$, and

(c) $t_0 + t_1 + t_2 + \cdots + t_{n-1} = 0 \pmod{2d+2}$.

1) Assume n is even. Let f be defined by the sequence $(t_0, t_1, \cdots, t_{n-1})$:

$$(d+1, \underbrace{d+2, d+2, \cdots, d+2}_{(\frac{n}{2}-1) \text{ terms}}, d+1, \underbrace{d, d, d, \cdots, d}_{(\frac{n}{2}-1) \text{ terms}}).$$

By Claim, it is easy to check that f is a circular $(2d+2)-$coloring for $C_n(d,1)$.

2) Let $n = 2k+1 \geq 9$ and $d < k$. Write $2k - d = 2m + m' + r'$, for some integers m, m', where $m' \in \{0,1\}$, and some real r', $0 \leq r' < 1$. Note, $m \geq 2$, as $k \geq 4$. Let $(t_0, t_1, t_2, \cdots, t_n)$ be:

$$(d+1, \underbrace{d+2, d+2, \cdots, d+2}_{m-1 \text{ terms}}, d+1, d, d+m', d, d+r', \underbrace{d, d, \cdots\cdots\cdots, d}_{2k-m-4 \text{ terms}}).$$

Because $k \geq 4$, it is straightforward to check that (a) $-$ (c) in the Claim are satisfied. We leave the details to the reader.

3) Assume $n = 7$ and $1 \leq d \leq 2$. Let $r' = 2 - d$. Then $0 \leq r' \leq 1$. Let $f : V(C_7) \to [0, 2d+2)$ be defined by the difference sequence: $(d, d+1, d+2, d+1, d, d+r', d)$. It is easy to check that f is a circular $(2d+2)-$coloring for $C_7(d,1)$. ∎

Lemma 13 *Assume* $\chi_c(C_n(d,1)) = p = 2d+1+r$ *for some real* $0 \leq r < 1$. *Let* f *be a circular* $p-$*coloring for* $C_n(d,1)$ *with* $f(v_0) = 0$ *and* $f(v_1) \leq p/2$. *Assume* $f(v_{i+1}) = f(v_i) + d + t_i \pmod{p}$ *for* $i = 0, 1, \cdots, n-1$. *Let* $t = t_0 + t_1 + \cdots + t_{n-1}$. *Then the following hold:*

(1) $0 \leq t_i < (1+r)/2$ *and* $t_i + t_{i+1} \leq r$, *for* $i = 0, 1, \cdots, n-1$.

(2) $0 \leq t \leq \frac{nr}{2}$ *and* $nd + t = 0 \pmod{p}$.

Proof. Let f be a circular $p-$labeling for $C_n(d,1)$, where $p = 2d+1+r$, $0 \leq r < 1$, $f(v_0) = 0$, and $f(v_1) \leq p/2$. Then $f(v_1) < p/2$. For if $f(v_1) = p/2$, then it is impossible to color v_2, as $v_1 \sim v_2$, $\text{dist}(v_2, v_0) = 2$, and $p/2 < d+1$. Hence, $d \leq l(f(v_0), f(v_1)) < p/2$ and $0 \leq t_0 < (1+r)/2$. Also, this implies that $d \leq l(f(v_1), f(v_2)) < p/2$. For if $l(f(v_1), f(v_2)) \geq p/2$, then as $\text{dist}(v_2, v_0) = 2$ and $\text{dist}(v_2, v_1) = 1$, we must have $1 \leq f(v_2) \leq f(v_1) - d$, which is impossible as $f(v_1) < p/2 < d+1$. Therefore, we conclude that $f(v_2) \leq p - 1 = 2d + r$. Indeed, this can be extended to that $f(v_{i+2}) \leq f(v_i) + 2d + r \pmod{p}$ for any i. Hence, $t_i + t_{i+1} \leq r$. This proves (1).

Because $v_0 = v_n$, by (1), we have $f(v_0) = f(v_n) = f(v_0)+nd+t \pmod p$, so $nd + t = 0 \pmod p$. If n is even, (2) follows by (1) immediately. If n is add, then fix a smallest t_j among all t_i's (so $t_j \le r/2$), and then pair up the rest of t_i's by $t_i + t_{i+1} \le r$. ∎

Let f be a circular $p-$coloring for a symmetric weighted graph $G(V, A)$. An edge (u, v) is said to be *tight* if $|f(u) - f(v)|_p = a_{uv}$. A cycle $C = (u_1, u_2, \cdots, u_m)$ is *tight* if all its edges (u_1, u_2), (u_2, u_3), $\cdots$, (u_m, u_1) are tight. If C is a tight cycle, then the *weight of* C, $a(C) = a_{u_1u_2} + a_{u_2u_3} + \cdots + a_{u_mu_1}$ is an integral multiple of p, the number $w(C) = \frac{a(C)}{p}$ is called the *winding number* of C. Mohar [9] proved that if $p = \chi_c(G)$, then there is a circular $p-$coloring of G which has a tight cycle.

Assume $n = 2k + 1$. For any $1/2 < d < k$, let $z(d) = \lfloor \frac{nd}{2d+1} \rfloor = k - h$ for some integer h. As $d < k$, it follows that $h \ge 1$. Because $nd = (2k + 1)d = (2d + 1)(k - h) + 2dh + d - k + h$, so $0 \le 2dh + d - k + h < 2d + 1$, implying

$$\frac{k-h}{2h+1} \le d < \frac{k-h+1}{2h-1}. \qquad (3.1)$$

Lemma 14 *Let $n = 2k + 1$. For any d,*

$$\chi_c(C_n(d, 1)) \le \min \{\frac{nd}{z(d)}, \frac{n}{2h-1}\}.$$

Proof. Let $p = \frac{nd}{z(d)}$. Then $f(v_i) = id \bmod p$ is a circular $p-$coloring for $C_n(d, 1)$. Let $p' = \frac{n}{2h-1}$. By (3.1), $d < \frac{k-h+1}{2h-1}$, so $f'(v_i) = (\frac{p'-1}{2})i \bmod p'$ is a circular $p'-$coloring for $C_n(d, 1)$. Note, $(v_0, v_1, v_2, \cdots, v_{n-1})$ is a tight cycle in f, and $(v_0, v_2, v_4, \cdots, v_{n-1}, v_1, v_2, \cdots, v_{n-1})$ is a tight cycle in f', with winding numbers, respectively, $z(d)$ and $2h - 1$. ∎

Theorem 15 *Let $n = 2k + 1$. If $\chi_c(C_n(d, 1)) < 2d + 2$, then*

$$\chi_c(C_n(d, 1)) = \min \{\frac{nd}{z(d)}, \frac{n}{2h-1}\}.$$

Proof. Assume $\chi_c(C_n(d,1)) = p = 2d+1+r$, with $0 \le r < 1$. By Lemma 14, it suffices to show that $p \ge \min\{\frac{nd}{z(d)}, \frac{n}{2h-1}\}$. Let f be a circular $p-$coloring for $C_n(d,1)$, with $f(v_0) = 0$. Without loss of generality (by symmetry), assume $f(v_1) \le p/2$. By Lemma 13, $f(v_{i+1}) = f(v_i) + d + t_i$ and $nd + t = 0 \pmod p$, where $t = t_0 + t_1 + \cdots t_{n-2}$. Hence,

$$nd + t = (2k+1)d + t = (2d+1+r)x, \text{ for some integer } x. \tag{3.2}$$

If $x \le z(d)$, then $\chi_c(C_n(d,1)) = 2d+1+r = \frac{nd+t}{x} \ge \frac{nd}{z(d)}$, and we are done. Assume $x \ge z(d)+1$. Because $t \le \frac{nr}{2}$, by (3.2), we have $t = (2d+1+r)x - (2k+1)d \le \frac{nr}{2}$. This implies that $r \ge \frac{(2d+1)x-(2k+1)d}{k-1/2-x}$. By some easy calculation, we get $\chi_c(C_n(d,1)) = 2d+1+r \ge \frac{n}{2h-1}$. ∎

For $n = 2k+1 \ge 9$, by Theorems 10, 11, 15, and Lemma 12, we express the value of $\chi_c(C_{2k+1}(d,1))$ as a continuous, piecewise linear function of d:

Corollary 16 *Let $n = 2k+1 \ge 9$. For any $d > 1/2$, let $z(d) = \lfloor \frac{nd}{2d+1} \rfloor$. Then*

$$\chi_c(C_n(d,1)) = \begin{cases} n, & \text{if } d \in [k-(1/2), k); \\ 2d+2, & \text{if } d \in [\frac{2k-2}{3}, k-(1/2)); \\ \frac{n}{2h-1}, & \text{if } d \in [\frac{k-h}{2h-1}, \frac{k-h+1}{2h-1}),\ h = 2, 3, \cdots, \lfloor \frac{k+1}{2} \rfloor; \\ \frac{n}{k}, & \text{if } d \in (0, 1/2]; \\ \frac{nd}{z(d)}, & \text{otherwise.} \end{cases}$$

Proof. Let $n = 2k+1$. If $d \ge k$, then $z(d) = k$. So, the result for $d \ge k$ and $d \le 1/2$ follows by Theorems 11 and 10, respectively. In the following, assume $1/2 < d < k$.

Let $z(d) = \lfloor \frac{nd}{2d+1} \rfloor = k-h$. Assume $\chi_c(C_n(d,1)) = p = 2d+1+r$ for some $0 \le r < 1$. By Theorem 15, (a) $2d+1+r = \frac{n}{2h-1}$ or (b) $2d+1+r = \frac{nd}{z(d)}$ holds. As $r < 1$, if (a) holds, then $d > \frac{n}{2(2h-1)} - 1$; if (b) holds, then $d < \frac{2(k-h)}{2h+1}$. Assume $h = 1$. By (3.1), one gets:

$$\chi_c(C_n(d,1)) = \begin{cases} n, & \text{if } d \in (k-(1/2), k); \\ \frac{nd}{z(d)}, & \text{if } d \in [\frac{k-1}{3}, \frac{2(k-1)}{3}). \end{cases}$$

By Lemma 12, $\chi_c(C_n(d,1)) = 2d+2$, if $d \in [\frac{2(k-1)}{3}, k-(1/2)]$. If $h \geq 2$, functions (a) and (b) intersect at $d = \frac{k-h}{2h-1}$. Moreover, (a) is greater than (b), if $\frac{k-h}{2h-1} < d < \frac{k-h+1}{2h-1}$. Note that, as $h \geq 2$, we have $\frac{2(k-h)}{2h+1} \geq \frac{k-h+1}{2h-1}$ and $\frac{n}{2(2h-1)} - 1 \leq \frac{k-h}{2h-1}$. By (3.1), the result follows. ∎

Theorem 17

$$\chi_c(C_5(d,1)) = \begin{cases} (5/2)d, & \text{if } d \geq 2; \\ 5, & \text{if } d \in [1,2); \\ 5d, & \text{if } d \in (1/2,1); \\ 5/2, & \text{if } 0 < d \leq 1/2. \end{cases}$$

Proof. Note, $\chi_c(C_5(d,1)) \geq \min\{5d, 5\}$, as $C_5(d,1)$ is a complete graph, so any circular coloring is one-to-one, and the separation between any two consecutive labels must be at least $\min\{d,1\}$. Hence, the result follows by Theorems 10 and 11, and the circular $5-$coloring $(0,1,2,3,4)$ and the circular $5d-$coloring $(0,d,2d,3d,4d)$ for (v_0,v_1,v_2,v_3,v_4), when $d \in [1,2)$ and $d \in (1/2,1)$, respectively. ∎

Lemma 18 *If* $2 \leq d \leq 3$ *and* $\chi_c(C_7(d,1)) < \min\{7, 3d\}$*, then*

$$\chi_c(C_7(d,1)) \geq (5/2)d + 1.$$

Proof. Assume $2 \leq d \leq 3$ and $\chi_c(C_7(d,1)) = p < \min\{7, 3d\}$. Let f be a circular $p-$coloring for $C_7(d,1)$ with $f(v_0) = 0$. For any i, because $p < 3d$, so $1 \leq |f(v_i) - f(v_{i+2})|_p < d$, implying one of the following holds:

$$|f(v_{i+2}) - f(v_{i-2})|_p \geq 2, \text{ or} \tag{3.3}$$

$$|f(v_{i+2}) - f(v_{i-2})|_p < d - 1. \tag{3.4}$$

Assume for all i, (3.4) holds. So, $|f(v_2) - f(v_5)|_p < d-1$. Without loss of generality, we assume $f(v_2), f(v_5) \in [1,d)$. By (3.4), $|f(v_6) - f(v_2)|_p < d-1$ and $|f(v_5) - f(v_1)|_p < d-1$, which is impossible as $v_6 \sim v_0$ and $v_1 \sim v_0$.

Hence, there exists some i such that (3.3) holds. By symmetry, we may assume that $i = 0$, $1 \leq f(v_2) < d$ and $p - d < f(v_5) \leq p-1$. Then

$f(v_1), f(v_6) \in (f(v_2), f(v_5))$. If $f(v_1) < f(v_6)$, then the ordering of the labels of $(v_0,\ v_2,\ v_1,\ v_6,\ v_5)$ on S_p gives $p \geq 2d+3 \geq 7$, a contradiction. (Using only the sub-index i of each v_i, we abbreviate the above contradiction by $(0,2,1,6,5) \Rightarrow 2d+3$.) Hence, $f(v_6) < f(v_1)$. So, the labels $0{=}f(v_0) < f(v_2) < f(v_6) < f(v_1) < f(v_5)$ divide S_p into five intervals, $I_1 = (f(v_0), f(v_2))$, $I_2 = (f(v_2), f(v_6))$, $\cdots$, and $I_5 = (f(v_5), f(v_0))$. By definition, $f(v_4) \notin I_5$ and $f(v_3) \notin I_1$.

Assume $f(v_4) \in I_2$. Because $|f(v_4) - f(v_2)|_p < d$, $|f(v_6) - f(v_4)|_p < d$, and $|f(v_3) - f(v_4)|_p \geq d$, so $f(v_3) \notin I_2$. If $f(v_3) \in I_3$, then $(0,2,4,3,1) \Rightarrow 2d+3$. If $f(v_3) \in I_4$, then $(0,2,4,6,1,3,5) \Rightarrow 7$. If $f(v_3) \in I_5$, then $(0,2,4,6,1,5,3) \Rightarrow 7$. Therefore $f(v_4) \notin I_2$. Symmetrically and similarly, one can show that $f(v_3), f(v_4) \notin I_4$ and $f(v_3) \notin I_2$. Note, it is impossible that $f(v_3), f(v_4) \in I_3$, as $l(I_3) < d$.

By symmetry, it suffices to consider two cases: (1) $f(v_3) \in I_5$ and $f(v_4) \in I_1$; and (2) $f(v_3) \in I_3$ and $f(v_4) \in I_1$. Indeed, they are "identical." In (1) and (2), the orderings of the labels on S_p are, respectively (starting at v_0 and v_6, respectively, and using only the sub-index i for each v_i), $(0,4,2,6,1,5,3)$ and $(6,3,1,5,0,4,2)$. By increasing each number in the latter case by 1, they become identical.

It suffices to consider (2). Let $l(f(v_0), f(v_4)) = x_1$ and $l(f(v_1),\ f(v_5)) = x_2$. Assume $x_2 \geq x_1$ (the proof for $x_2 \leq x_1$ is similar). Set $l(f(v_4), f(v_2)) = 1+t_1$, $l(f(v_2), f(v_3)) = d+t_2$, $l(f(v_3), f(v_1)) = 1+t_3$, $l(f(v_5), f(v_4)) = d+t_4$, for some $t_1, t_2, t_3, t_4 \geq 0$. Then, $p = 2d+2+x_2+t_1+t_2+t_3+t_4$.

Because $v_0 \sim v_6$, we have $x_1 + 1 + t_1 +\ l(f(v_2), f(v_6)) \geq d$, implying $l(f(v_2), f(v_6)) \geq d - x_1 - 1 - t_1$. Similarly, because $v_6 \sim v_5$, we have $l(f(v_6), f(v_3)) \geq d - x_2 - 1 - t_3$. Therefore, $l(f(v_2), f(v_3)) = d + t_2 \geq 2d - 2 - x_1 - x_2 - t_1 - t_3$. Because $x_2 \geq x_1$, we conclude that $x_2 \geq (d-2)/2$, so $p \geq (5/2)d + 1$. ∎

Theorem 19

$$\chi_c(C_7(d,1)) = \begin{cases} (7/3)d, & \textit{if } d \geq 3; \\ 7, & \textit{if } 12/5 \leq d \leq 3; \\ (5/2)d+1, & \textit{if } 2 \leq d \leq 12/5; \\ 2d+2, & \textit{if } 4/3 \leq d \leq 2; \\ (7/2)d, & \textit{if } 2/3 \leq d \leq 4/3; \\ 7/3, & \textit{if } 0 < d \leq 2/3. \end{cases}$$

Proof. The result follows by Lemmas 12, 18, Theorems 10, 11, 15, and the circular $(\frac{5}{2}d+1)$−coloring $(0, d, 2d, \frac{d}{2}-1, \frac{3d}{2}, \frac{d}{2}, \frac{3d}{2}+1)$ for $C_7(d,1)$, when $2 \leq d \leq 12/5$. ∎

Similar to Theorem 15 and Corollary 16, we obtain the following result for even cycles. We leave the details to the reader.

Theorem 20 *Let $n = 2k$, and for any real $d > 1/2$, let $z(d) = \lfloor \frac{nd}{2d+1} \rfloor = k - h$. Then $\chi_c(C_n(d,1)) = \min\{\frac{nd}{z(d)}, \frac{k}{h-1}, 2d+2\}$. Or equivalently,*

$$\chi_c(C_n(d,1)) = \begin{cases} 2d+2, & \textit{if } d \geq k-1; \\ \frac{k}{m}, & \textit{if } d \in [\frac{k-m-1}{2m}, \frac{k-m}{2m}),\ m = 1, 2, \cdots, \lfloor \frac{k-1}{2} \rfloor; \\ 2, & \textit{if } k \textit{ is even and } d \in (0, 1/2]; \\ \frac{2k}{k-1}, & \textit{if } k \textit{ is odd and } d \in (0, 1/2]; \\ \frac{nd}{z(d)}, & \textit{otherwise.} \end{cases}$$

According to Lemma 2, we have determined the value of $\chi_c(C_n(d_1, d_2))$ for any positive reals d_1 and d_2. A special case of interest is that when $d_1 = d \times d_2$ for some integer d, then it bounds to find the value of $\chi_c(C_n(d,1))$. By Theorems 16, 17, 19, 20, and Corollary 16, we have:

Corollary 21 *Let d and n be positive integers, $n \geq 3$. Then*

$$\chi_c(C_n(d,1)) = \min\{2d+2, \frac{nd}{\lfloor \frac{nd}{2d+1} \rfloor}\}.$$

Corollary 21 implies that the value of $\chi_c(C_n(d,1))$, when d is an integer, can be as close as possible to the lower bound $\lceil \chi_c(C_n(d,1)) \rceil - 1$ (cf. (1.1)).

Note that, if $\chi_c(C_n(d,1)) = p < 2d+2$, then there exists a circular $p-$coloring for $C_n(d,1)$ with a tight cycle that has all edges of the same weight. (If n is odd, by the proof of Lemma 14, $(v_0, v_1,\ v_2, \cdots, v_{n-1})$ or $(v_0, v_2,\ v_4, \cdots, v_{n-1}, v_1,\ v_3,\ \cdots,\ v_{n-1})$ is a tight cycle; if n is even, then $(v_0, v_1,\ v_2, \cdots, v_{n-1})$ or $(v_0, v_2,\ v_4, \cdots, v_{n-2})$ is a tight cycle.) This is not the case, however, when $\chi_c(C_n(d,1)) \geq 2d+2$. For instance, consider $C_7(d,1)$ with $2 \leq d \leq 12/5$, it is impossible to get a circular $p-$coloring, $p = (5/2)d+1$, with a tight cycle that has all edges of the same weight. Indeed, one can get a circular $p-$coloring with the tight cycle $(v_0, v_1,\ v_2, v_3,\ v_5, v_4,\ v_6)$ of winding number 2.

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