Topic 9 & 10

Cauchy's Theorem

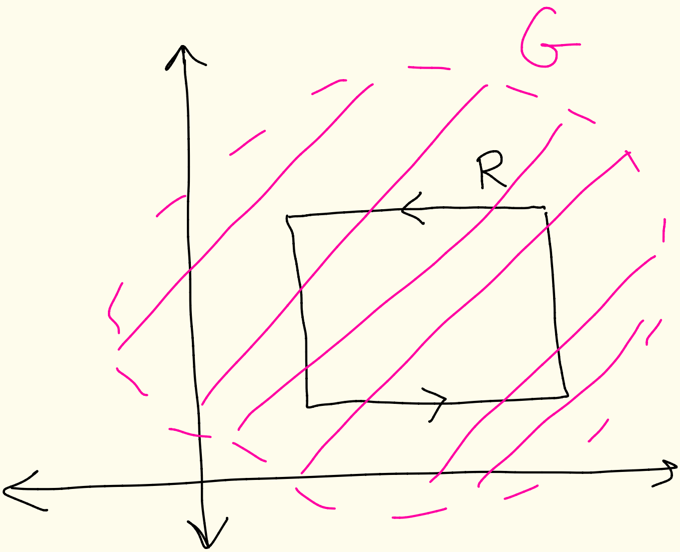
and

Cauchy's Integral Formula

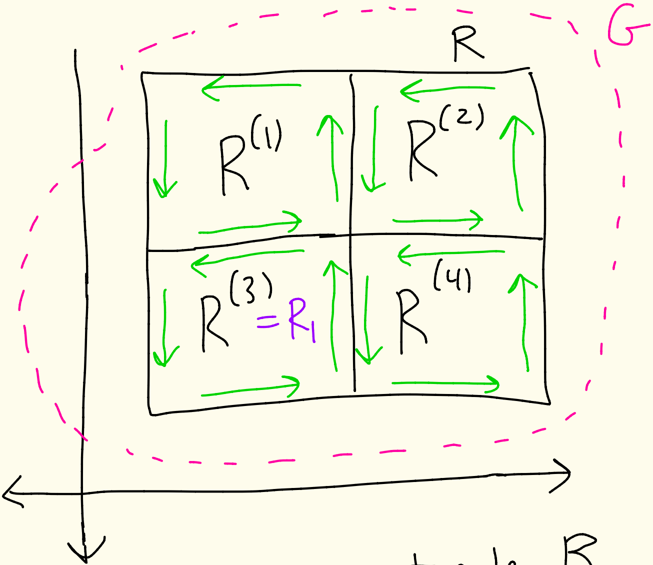
<u>Theorem:</u> (Cauchy's theorem for a rectangle)
Suppose that R is a rectangular path with sides parallel to the xy-axes and that f is a function defined and analytic on an open set G containing R and its interior.

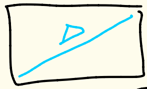
Then, $$\int_R f = 0$$

The γ here is R



Orient R in counter-clockwise direction.





Let P be the perimeter of R [ie length of R].

Let Δ be the length of R's main diagonal.

Divide the rectangle R into four congruent smaller rectangles $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, and $R^{(4)}$

If each subrectangle is oriented in the counter-clockwise direction, then the cancellation along common edges gives

$$\int_R f = \int_{R^{(1)}} f + \int_{R^{(2)}} f + \int_{R^{(3)}} f + \int_{R^{(4)}} f$$

Since $\left|\int_R f\right| \leq \left|\int_{R^{(1)}} f\right| + \left|\int_{R^{(2)}} f\right| + \left|\int_{R^{(3)}} f\right| + \left|\int_{R^{(4)}} f\right|$

there must be at least one of the rectangles for which $\left|\int_{R^{(k)}} f\right| \geq \frac{1}{4}\left|\int_R f\right|$. Call this $R^{(k)}$ by R_1

③

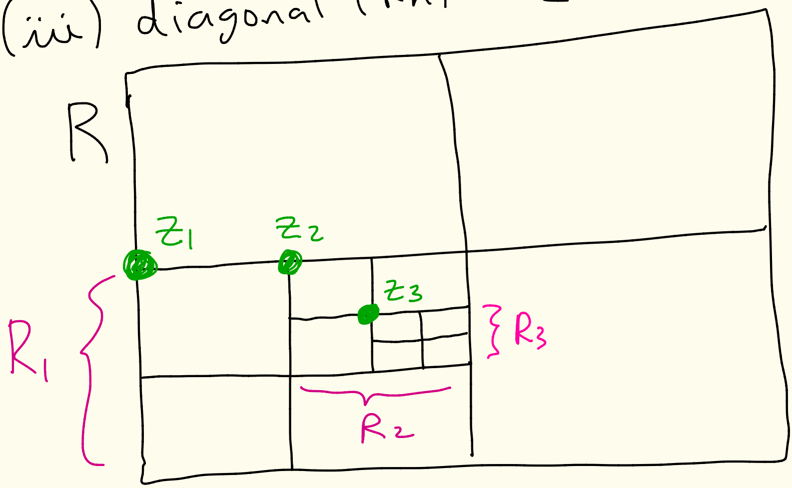
Notice that the perimeter and diagonal of R_1 are half the perimeter and diagonal of R.

Now repeat this bisection process inside of R_1, obtaining a sequence $R_1, R_2, R_3, \ldots$ of smaller and smaller rectangles such that

(i) $\left|\int_{R_n} f\right| \geq \frac{1}{4}\left|\int_{R_{n-1}} f\right| \geq \cdots \geq \frac{1}{4^n}\left|\int_R f\right|$

(ii) $\text{Perimeter}(R_n) = \frac{1}{2^n}\,\text{perimeter}(R) = \frac{P}{2^n}$

(iii) $\text{diagonal}(R_n) = \frac{1}{2^n}\,\text{diagonal}(R) = \frac{\Delta}{2^n}$



Let z_n be the upper left-corner of R_n

<u>Claim:</u> (z_n) is a Cauchy sequence.

<u>pf of claim:</u> Let $\varepsilon > 0$. Choose $N > 0$ such that $\dfrac{\Delta}{2^N} < \varepsilon$.

If $n, m \geq N$, then

$$|z_n - z_m| \leq \text{diagonal}(R_N) = \frac{\Delta}{2^N} < \varepsilon.$$

z_n, z_m are in or on R_N

$\boxed{\text{Claim}}$

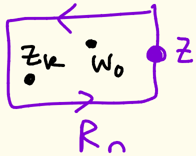
Therefore, there exists $w_0 \in \mathbb{C}$ with $\lim\limits_{n \to \infty} z_n = w_0$.

$\boxed{\text{Claim}}$

$\boxed{\text{Let } \varepsilon > 0 \text{ be fixed for the remainder of the proof.}}$

We need several facts:

Fact 0: If z is on R_n, then $|z - w_0| \leq \frac{\Delta}{2^n}$



pf: Let z be on R_n.
If $k \geq n$, then z_k is on or inside R_n.
Since $R_n \cup (\text{int of } R_n)$ is a closed set and $\lim_{k \to \infty} z_k = w_0$, then by HW 8 #6 we have that w_0 is on or inside of R_n.
Since z is on R_n and w_0 is on or inside R_n, they can't be further apart than the diagonal of R_n. Thus, $|z - w_0| \leq \frac{\Delta}{2^n}$.

$\boxed{\text{Fact 0}}$

Fact 1: w_0 lies on or inside R, so $f'(w_0)$ exists.
pf: This follows from the proof of Fact 0.

$\boxed{\text{Fact 1}}$

Fact 2: Since $f'(w_0)$ exists, there exists $\delta > 0$ such that if $0 < |z - w_0| < \delta$ then $\left| \frac{f(z) - f(w_0)}{z - w_0} - f'(w_0) \right| < \varepsilon$

Thus, if $|z - w_0| < \delta$, then

$$|f(z) - f(w_0) - f'(w_0)(z - w_0)| \leq \varepsilon |z - w_0|$$

Let $\hat{N}$ be large enough so that if $n \geq \hat{N}$ then $\frac{\Delta}{2^n} < \delta$.

Thus, if $n \geq \hat{N}$ and z is on R_n then by Fact 0 we have $|z - w_0| < \frac{\Delta}{2^n} < \delta$.

Thus, if $n \geq \hat{N}$ and z is on R_n then

$$|f(z) - f(w_0) - f'(w_0)(z - w_0)| \leq \varepsilon |z - w_0| < \frac{\varepsilon \Delta}{2^n}.$$

Fact 2

Fact 3: By FTOC,

$$\int_{R_n} 1\, dz = 0 \quad \text{and} \quad \int_{R_n} (z-w_0)\, dz = 0$$

because R_n is a closed curve. $\boxed{\text{Fact 3}}$

Thus if $n \geq \hat{N}$ we have that

$$\left| \int_R f \right| \leq 4^n \left| \int_{R_n} f \right|$$

$$\overset{\text{Fact 3}}{=} 4^n \left| \int_{R_n} f(z)\, dz - f(w_0) \overbrace{\int_{R_n} 1\, dz}^{0} - f'(w_0) \underbrace{\int_{R_n} (z-w_0)\, dz}_{0} \right|$$

$$= 4^n \left| \int_{R_n} [f(z) - f(w_0) - f'(w_0)(z-w_0)]\, dz \right|$$

$$\overset{\text{Fact 2}}{\leq} 4^n \left(\frac{\varepsilon \Delta}{2^n} \right) \underbrace{\text{perimeter}(R_n)}_{\text{arclength}(R_n)} = \varepsilon \Delta 2^n \cdot \frac{P}{2^n}$$

$$= \varepsilon \Delta P$$

So, $\left| \int_R f \right| \leq \varepsilon \underbrace{\Delta P}_{\text{fixed \#}}$

for all $\varepsilon > 0$.

Thus, $\left| \int_R f \right| = 0$.

So, $\int_R f = 0$ $\blacksquare$

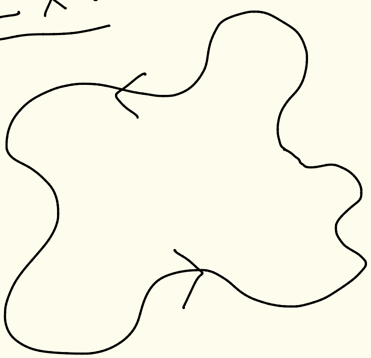
Time to generalize Cauchy!

closed means:
$\gamma(a) = \gamma(b)$

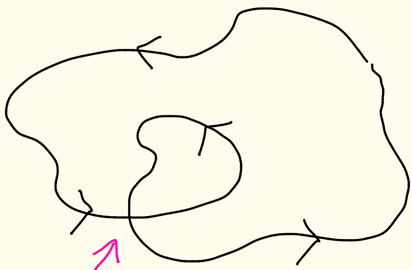
Def: A closed curve $\gamma : [a,b] \to \mathbb{C}$ is called simple if only the initial and final values ($\gamma(a)$ & $\gamma(b)$) are the same.

[Ie, if $\gamma(t_1) = \gamma(t_2)$ then $t_1, t_2 \in \{a, b\}$]

Ex: γ is simple



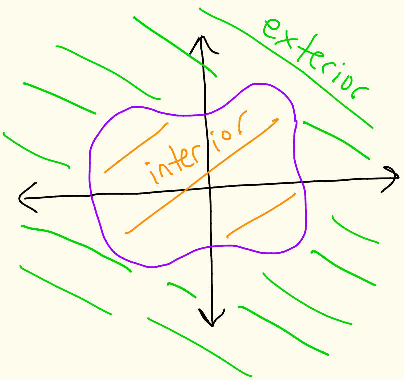
γ is not simple

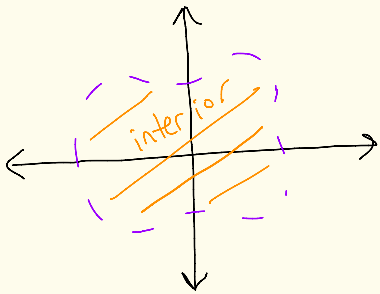


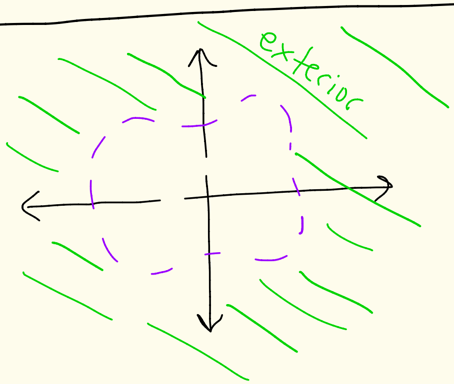
makes it not simple

Jordan Curve Theorem

Every simple closed curve in the complex plane divides the plane into two disjoint open sets. One set (the interior of the curve) is open and bounded and the other (the exterior of the curve is open and unbounded.

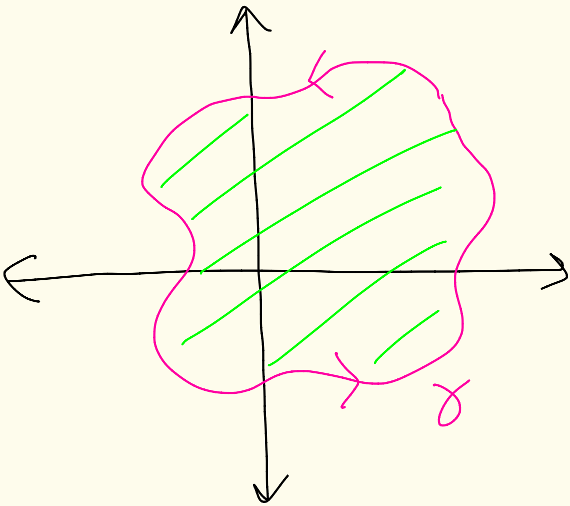






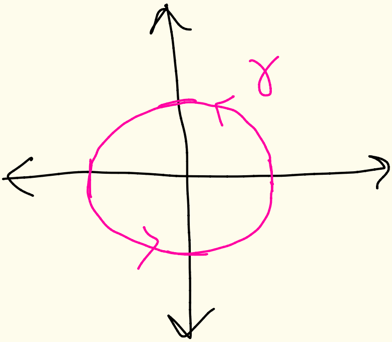
Generalization of Cauchy's Theorem on a rectangle

<u>Thm:</u> (Cauchy's Thm) If a function f is analytic at all points interior to and on a simple closed piece-wise smooth curve γ, then $\int_\gamma f = 0$.



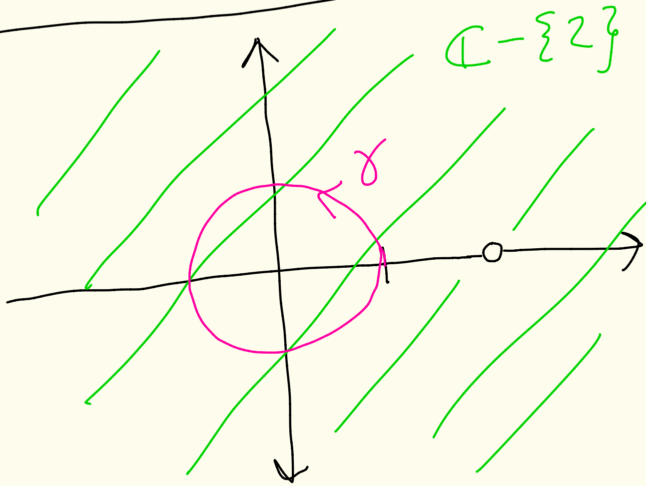
<u>Note:</u> Recall that f is analytic at a point z_0 means that f is differentiable in a r-neighborhood of z_0. Thus, really this thm is assuming f is analytic on an open set containing γ and its interior.

Ex: Let γ be the unit circle oriented counterclockwise.



The function $\frac{1}{z-2}$ is analytic on $\mathbb{C}-\{2\}$. So, $\frac{1}{z-2}$ is analytic on γ and inside γ so by Cauchy's Thm

$$\int_\gamma \frac{dz}{z-2} = 0$$



<u>Theorem:</u> Suppose that γ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are simple, closed, piecewise smooth curves such that

(a) γ is oriented in the counterclockwise direction

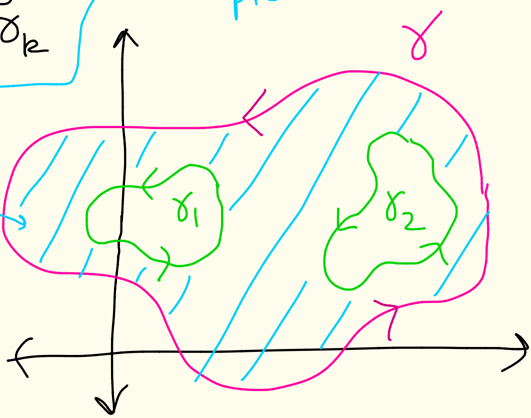
(b) $\gamma_1, \gamma_2, \ldots, \gamma_n$ are all oriented in the counterclockwise direction, are all interior to γ, and the interiors of $\gamma_1, \gamma_2, \ldots, \gamma_n$ have no points in common.

If f is analytic throughout the closed set consisting of all the points within and on γ except for points interior to any γ_k, then

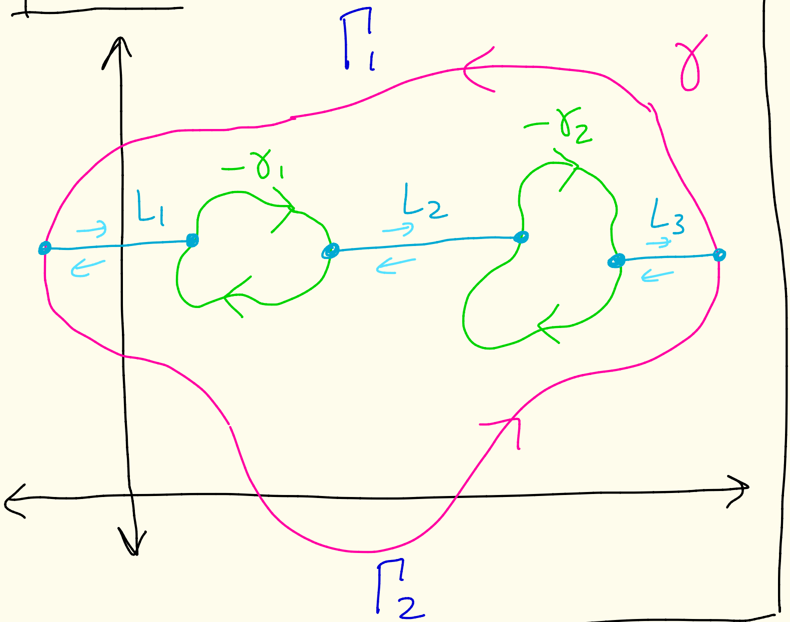
$$\int_\gamma f = \sum_{k=1}^{n} \int_{\gamma_k} f$$

picture with $n=2$

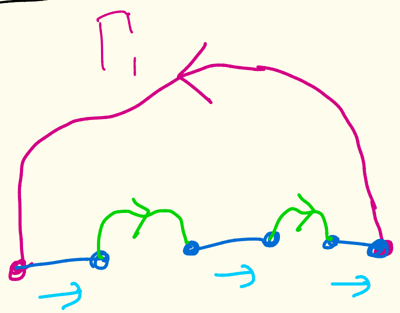
f is analytic on γ, between γ and γ_k and on γ_k

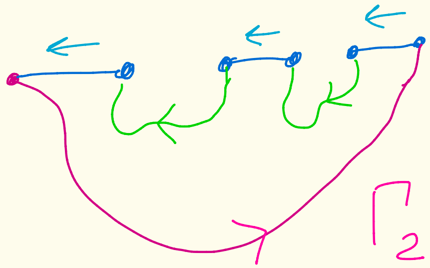


proof:



Make the lines $L_1, L_2, \ldots, L_n$ as in the picture above. Let Γ_1 be the top curve and Γ_2 be the bottom curve.





By Cauchy's Theorem,

$$\int_{\Gamma_1} f = 0 \quad \text{and} \quad \int_{\Gamma_2} f = 0.$$

means go over Γ_1 and then go over Γ_2

Thus,

$$0 = \int_{\Gamma_1} f + \int_{\Gamma_2} f = \int_{\Gamma_1 + \Gamma_2} f$$

$$= \int_{\gamma} f + \sum_{k=1}^{n} \int_{-\gamma_k} f$$

$$= \int_{\gamma} f - \sum_{k=1}^{n} \int_{\gamma_k} f.$$

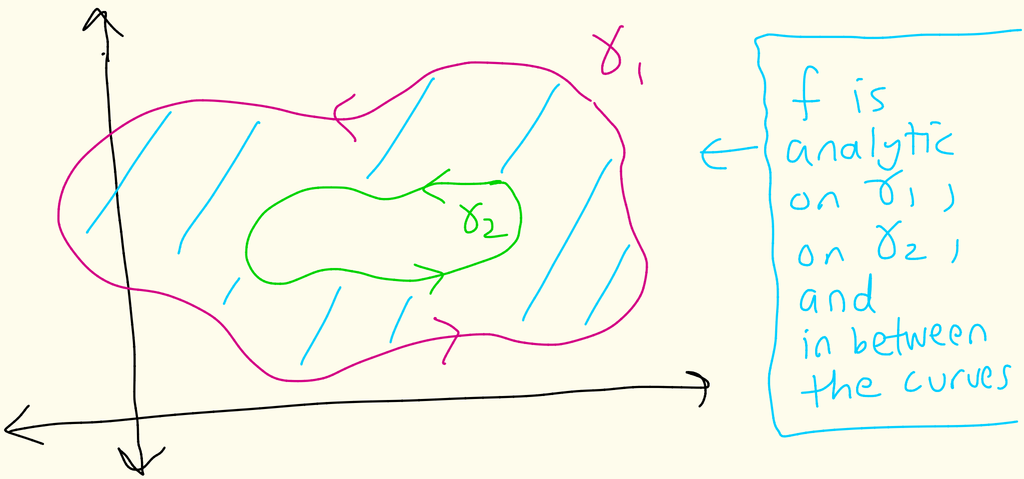
Thus,

$$\int_{\gamma} f = \sum_{k=1}^{n} \int_{\gamma_k} f$$

$\blacksquare$

Corollary: Let γ_1 and γ_2 be simple, closed, piecewise smooth curves oriented in the counterclockwise direction. Assume γ_2 is interior to γ_1. If f is analytic in the closed set consisting of all points on γ_1, on γ_2, and all points between them, then

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$



This Corollary is known as the principal of deformation of paths since it tells us that if γ_1 is continuously deformed into γ_2, always passing through points at which f is analytic, then the value of the integral of f doesn't change as γ_1 deforms to γ_2.

This comes up later when you want to generalize these theorems using something called homotopy.

Theorem (Cauchy Integral Formula)

Let f be analytic everywhere within and on a simple, closed, piece-wise smooth curve γ, taken in the counter-clockwise direction. If z_0 is any point interior to γ, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

$\boxed{\text{Ex:}}$ Let γ be the circle $|z| = 2$ oriented in the counter-clockwise direction.

$$\int_\gamma \frac{z}{(9-z^2)(z+i)}\, dz = \int_\gamma \frac{\left(z/9-z^2\right)}{z-(-i)}\, dz$$

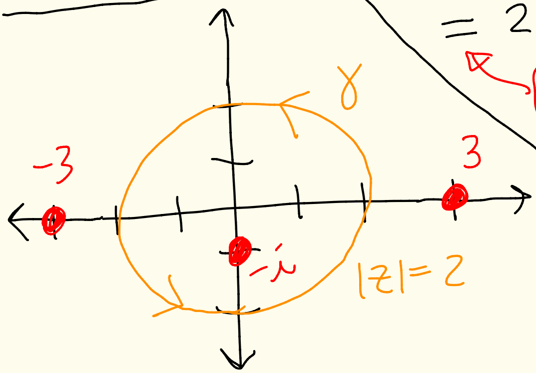
$f(z) = \dfrac{z}{9-z^2}$ is analytic on and inside γ

$$= 2\pi i\, f(-i)$$

Cauchy integral thm

$$= 2\pi i \left(\frac{-i}{9-(-i)^2}\right)$$

$$= \pi/5$$



$\dfrac{z}{(9-z^2)(z+i)}$ is analytic everywhere except at 3, -3, and $-i$

proof of Cauchy integral formula:

Let z_0 be a point interior to γ.

Let $\varepsilon > 0$.

Since f is continuous at z_0 there exists $\delta > 0$ where

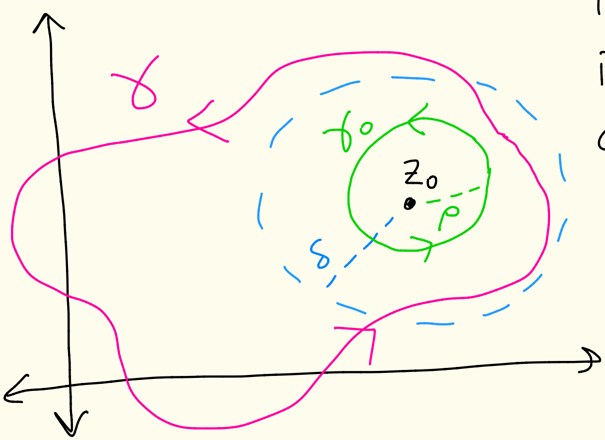
$$|f(z) - f(z_0)| < \varepsilon$$

whenever $|z - z_0| < \delta$ and z is in the domain of f.

because $\lim_{z \to z_0} f(z) = f(z_0)$

By the Jordan-curve theorem, the interior of γ is open.

Thus, there exists $\rho > 0$ such that the circle $|z - z_0| = \rho$ is interior to γ.



Choose ρ such that $\rho < \delta$.

Let γ_0 denote the circle $|z - z_0| = \rho$ oriented counterclockwise.

Since $\dfrac{f(z)}{z-z_0}$ is analytic on γ, in between γ and γ_0, and on γ_0, by the Corollary to Cauchy's theorem we have

$$\int_{\gamma} \frac{f(z)}{z-z_0}\,dz = \int_{\gamma_0} \frac{f(z)}{z-z_0}\,dz$$

Thus,

$$\begin{aligned}
&\int_{\gamma} \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) \\
&= \underbrace{\int_{\gamma_0} \frac{f(z)}{z-z_0}\,dz}_{\text{from above}} - f(z_0) \underbrace{\int_{\gamma_0} \frac{dz}{z-z_0}}_{\substack{= 2\pi i \\ \text{from a previous class}}} \\
&= \int_{\gamma_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz
\end{aligned} \tag{*}$$

One can show that the arclength of γ_0 is $2\pi\rho$.

Thus,

$$\left| \int_{\gamma_0} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| < \frac{\varepsilon}{\rho} \cdot \underbrace{2\pi\rho}_{\text{arclength of } \gamma_0} = 2\pi\varepsilon$$

If z is on γ_0 then

$|f(z)-f(z_0)| < \varepsilon$

$|z-z_0| = \rho$

So,

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| \leq \frac{\varepsilon}{\rho}$$

on γ_0

So, by $(*)$ $\left| \int_{\gamma} \frac{f(z)}{z-z_0}\, dz - 2\pi i f(z_0) \right| < 2\pi\varepsilon$

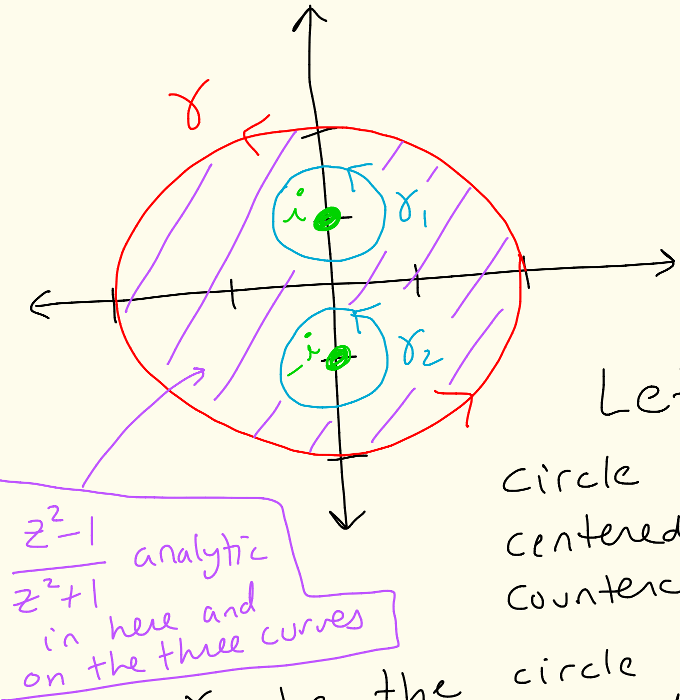
for every $\varepsilon > 0$. So,

$$\int_{\gamma} \frac{f(z)}{z-z_0}\, dz - 2\pi i f(z_0) = 0.$$

$\blacksquare$

Ex: Let's calculate

$$\int_\gamma \frac{z^2-1}{z^2+1}\,dz$$ where γ is the circle of radius 2 centered at 0.



$\frac{z^2-1}{z^2+1}$ is analytic everywhere except at $z = \pm i$.

Let γ_1 be the circle of radius $1/2$ centered at i, oriented counterclockwise.

Let γ_2 be the circle of radius $1/2$ centered at $-i$, oriented counterclockwise.

Notice that $\frac{z^2-1}{z^2+1}$ is analytic on γ, γ_1, γ_2, and in-between the curves.

So, by a corllary to Cauchy's thm

$$\int_\gamma \frac{z^2-1}{z^2+1}\,dz = \int_{\gamma_1} \frac{z^2-1}{z^2+1}\,dz + \int_{\gamma_2} \frac{z^2-1}{z^2+1}\,dz$$

$$= \int_{\gamma_1} \frac{\left(\frac{z^2-1}{z+i}\right)}{z-i}\,dz + \int_{\gamma_2} \frac{\left(\frac{z^2-1}{z-i}\right)}{z+i}$$

numerator is analytic in and on γ_1

numerator analytic in and on γ_2

$z-(-i)$

$$\boxed{\frac{z^2-1}{z^2+1} = \frac{z^2-1}{(z+i)(z-i)}}$$

$$= 2\pi i\left[\frac{i^2-1}{i+i}\right] + 2\pi i\left[\frac{(-i)^2-1}{(-i)-i}\right]$$

$\boxed{\text{Cauchy integral formula}}$

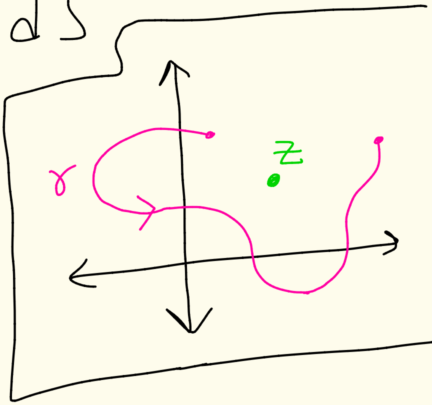
$$= 2\pi i\left[\frac{-1-1}{2i} + \frac{-1-1}{-2i}\right]$$

$$= \pi[-2+2] = 0$$

Theorem: Let γ be a piecewise smooth curve in $\mathbb{C}$. Let g be continuous on γ. Define $G : \mathbb{C} - \gamma \to \mathbb{C}$ by

$$G(z) = \int_{\gamma} \frac{g(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in \mathbb{C} - \gamma$.



Then, G is analytic on $\mathbb{C} - \gamma$ and

$$G'(z) = \int_{\gamma} \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta$$

for all $z \in \mathbb{C} - \gamma$.

Also in Hoffman/marsden book but more general

From: Complex Analysis Man Wah Wong

proof (a little sketchy):

Let $z \in \mathbb{C} - \gamma$.

We need to show that

$$\underbrace{\lim_{h \to 0} \frac{G(z+h) - G(z)}{h}}_{G'(z)} = \int_\gamma \frac{g(\xi)}{(\xi - z)^2} \, d\xi$$

$h \in \mathbb{C}$

$$\boxed{G(z) = \int_\gamma \frac{g(\xi)}{\xi - z} \, d\xi}$$

We have that

$$\frac{G(z+h) - G(z)}{h} = \frac{1}{h} \int_\gamma \left(\frac{1}{\xi - (z+h)} - \frac{1}{\xi - z} \right) g(\xi) \, d\xi$$

$$= \int_\gamma \frac{g(\xi)}{[\xi - z - h][\xi - z]} \, d\xi$$

$$\frac{1}{\xi - (z+h)} - \frac{1}{\xi - z} = \frac{\xi - z - (\xi - z - h)}{(\xi - z - h)(\xi - z)} = \frac{h}{(\xi - z - h)(\xi - z)}$$

So,

$$\frac{G(z+h) - G(z)}{h} - \int_\gamma \frac{g(\zeta)}{(\zeta - z)^2} \, d\zeta$$

$$= \int_\gamma \frac{g(\zeta)}{(\zeta - z - h)(\zeta - z)} \, d\zeta - \int_\gamma \frac{g(\zeta)}{(\zeta - z)^2} \, d\zeta$$

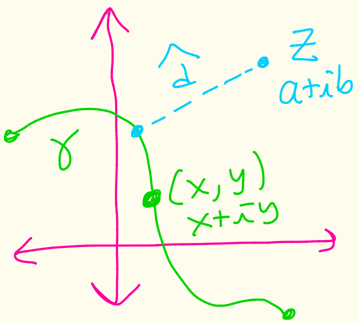
$$= \int_\gamma \frac{g(\zeta)[\zeta - z] - g(\zeta)[\zeta - z - h]}{(\zeta - z - h)(\zeta - z)^2} \, d\zeta$$

$$= \boxed{h \int_\gamma \frac{g(\zeta)}{(\zeta - z - h)(\zeta - z)^2} \, d\zeta = J_h(z)}$$

We are going to bound $|J_n(z)|$.

Let $\hat{d}$ be the distance between z and γ.

Why can we define $\hat{d}$?



Let $z = a+ib$.

By HW 8 #7, γ is a closed set in $\mathbb{C}$.

Because $\gamma : [a,b] \to \mathbb{C}$ is continuous on the closed and bounded set $[a,b]$, the image is also closed and bounded [topology].

So, γ in $\mathbb{C}$ is closed and bounded.

The function $d : \gamma \to \mathbb{R}$ defined by

$$d(x,y) = \sqrt{(x-a)^2 + (y-b)^2}$$

is continuous on γ which is closed and bounded. So, d has a minumum on γ, call it $\hat{d}$. $\boxed{\text{Why?}}$

Since we are going to let $h \to 0$ we can assume $|h| < \frac{\hat{d}}{2}$.

Then on γ, (ie ρ is on γ) we have

$$|\rho - z - h| \geq \big||\rho - z| - |h|\big|$$

$|c-d| \geq \big||c|-|d|\big|$

$\rho \in \gamma$, so $|\rho - z| \geq \hat{d} > |h|$

$$= |\rho - z| - |h|$$

$|\rho - z| \geq \hat{d}$
$|h| < \frac{\hat{d}}{2}$
$-|h| > -\frac{\hat{d}}{2}$

$$\geq \hat{d} - \frac{\hat{d}}{2} = \frac{\hat{d}}{2}$$

Since g is continuous on the closed and bounded set γ, g has a maximum on γ. Thus, there exists $M > 0$ where $|g(\rho)| \leq M$ for all ρ on γ.

Topology/metric spaces stuff

Continuous functions on closed/bounded sets have max/min on that set

Thus if ς is on γ then

$$\left|\frac{g(\varsigma)}{(\varsigma-z-h)(\varsigma-z)^2}\right| \leq \frac{2M}{\hat{d}^3}$$

$$\boxed{\begin{aligned} &|g(\varsigma)| \leq M \\ &|\varsigma-z-h| \geq \frac{\hat{d}}{2} \\ &|\varsigma-z| \geq \hat{d} \\ &\frac{1}{|\varsigma-z-h||\varsigma-z|^2} \leq \frac{2}{\hat{d}\,\hat{d}^2} \end{aligned}}$$

If L is the arclength of γ, then

$$|J_h(z)| = \left| h\int_\gamma \frac{g(\varsigma)}{(\varsigma-z-h)(\varsigma-z)^2}\,d\varsigma \right|$$

$$\leq |h| \cdot \frac{2M}{\hat{d}^3} \cdot L \longrightarrow 0$$

as $h \to 0$.

constants

Thus,

$$\lim_{h \to 0} \frac{G(z+h) - G(z)}{h} = \int_\gamma \frac{g(\zeta)}{(\zeta - z)^2} \, d\zeta, \quad \blacksquare$$

<u>Remark</u> (See Hoffman/Marsden for a proof)

With the same setup as the theorem it can be proved that if $z \in \mathbb{C} - \gamma$ then

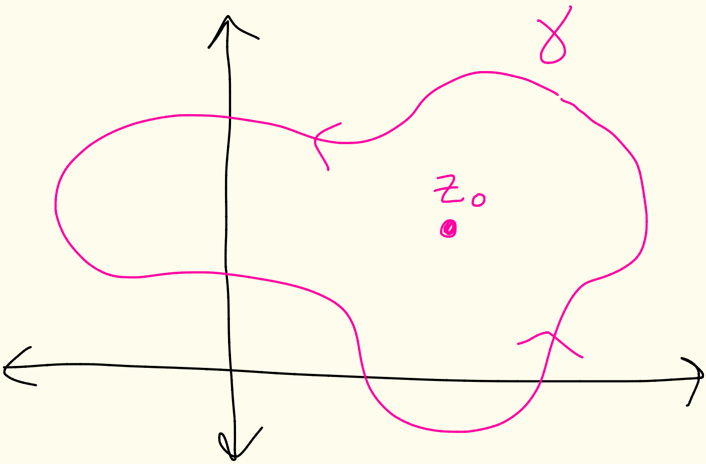
$$G^{(k)}(z) = k! \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta$$

for $k = 1, 2, 3, \ldots$

Theorem: (Cauchy Integral Thm)

Let f be analytic everywhere within and on a simple, closed, piece-wise smooth curve γ, oriented counterclockwise. If z_0 is any point interior to γ, then f is infinitely differentiable at z_0 and

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$



pf: Let z_0 be interior to γ.

By the previous Cauchy integral thm that we proved

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)} \, dz$$

By the thm we just proved

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^2} \, dz$$

because f is continuous on γ [because f is analytic on γ]

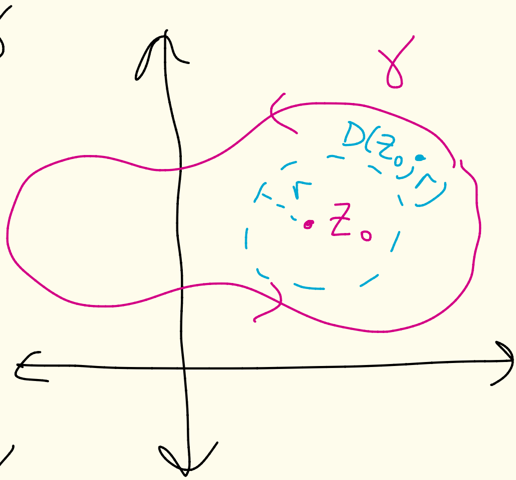
Use the remark to get

$$f''(z_0) = \frac{2!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^3} \, dz$$

You can keep using the remark over and over to get the general formula for $f^{(k)}(z_0)$

Note that this formula applies to all z_0 interior to γ.

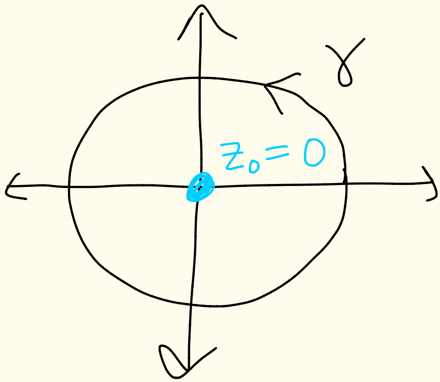
Thus, if you pick a specific z_0 interior to γ we can put an open disc of some radius r around z_0 that is completely inside γ



[because the interior of γ is open by the Jordan curve thm]

By what we just did, f is infinitely differentiable in $D(z_0;r)$. So, $f^{(k)}$ are analytic at z_0 ∎

Ex: Let γ be the unit circle, oriented counterclockwise.



$f(z) = e^{2z}$ | $z_0 = 0$

$$\int_\gamma \frac{e^{2z}}{z^4}\,dz = \int_\gamma \frac{e^{2z}}{(z-0)^4}\,dz$$

$f^{(3)}(z) = 2^3 e^{2z}$

$$= \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{3!} \cdot 8e^{2(0)}$$

$$= \frac{16\pi i}{6} = \frac{8\pi i}{3}$$