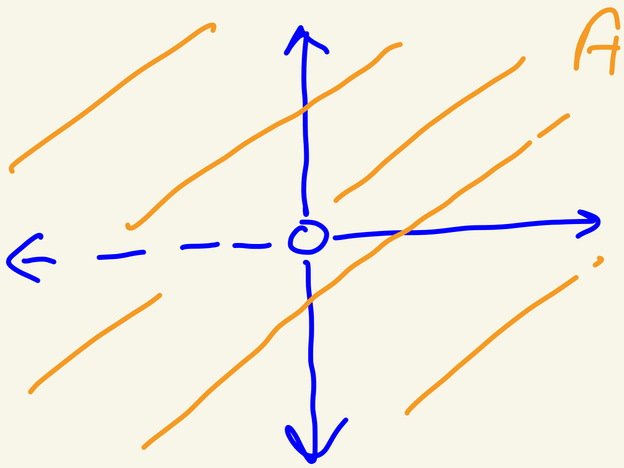
① (a)

$$f(z) = z^{2+3i} = e^{(2+3i)\log(z)}$$

$f'(z)$ exists where $\log(z)$ is analytic.

This is on $A = \mathbb{C} - \{x+iy \mid x \leq 0 \text{ and } y=0\}$



For $z \in A$ we have that

$$f'(z) = (2+3i)z^{(2+3i)-1} = (2+3i)z^{1+3i}$$

① (b)

$$f(x+iy) = \underbrace{(-x^2+2x^3-6xy^2)}_{u(x,y)} + i\underbrace{(6x^2y-2y^3)}_{v(x,y)}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -2x+6x^2-6y^2 \\ \frac{\partial v}{\partial y} &= 6x^2-6y^2 \end{aligned}\right\} \quad -2x+6x^2-6y^2 = 6x^2-6y^2$$

iff

$-2x = 0$

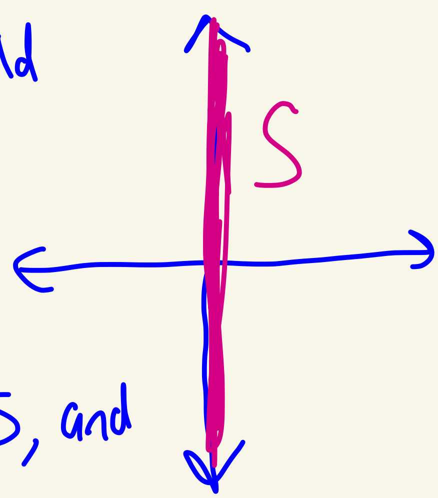
iff

$x = 0$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= -12xy \\ -\frac{\partial v}{\partial x} &= -(12xy) = -12xy \end{aligned}\right\} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ everywhere}$$

Let $S = \{x+iy \mid x=0, y\in\mathbb{R}\} = \{iy \mid y \in \mathbb{R}\}$

The Cauchy-Riemann equations hold only on S. And $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ exist and are continuous on all of $\mathbb{C}$, ie on an open set containing S.



Thus, $f'(z)$ exists for all $z \in S$, and

$$f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (-2x+6x^2-6y^2) + i(12xy) \text{ on } S$$

2. $\gamma(t) = 2 + t\left[(3+3i) - 2\right], \ 0 \le t \le 1$

$\gamma(t) = 2 + t(1+3i), \ 0 \le t \le 1$

$\gamma(t) = (2+t) + i(3t), \ 0 \le t \le 1$

$\gamma'(t) = 1 + 3i$

$$\int_\gamma \bar{z}\, dz = \int_0^1 \underbrace{\overline{\left[(2+t)+i(3t)\right]}}_{\bar{z} \text{ with } \gamma \text{ plugged in}} \underbrace{(1+3i)}_{\gamma'(t)}\, dt$$

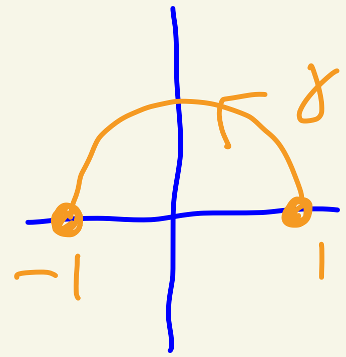
$$= \int_0^1 \left[(2+t) - 3it\right](1+3i)\, dt$$

$$= \int_0^1 (2 + 6i + t + 3it - 3it + 9t)\, dt$$

$$= \int_0^1 \left[(2+6i) + 10t\right] dt$$

$$= (2+6i)t + 5t^2 \Big|_0^1 = (2+6i) + 5 = \boxed{7+6i}$$

③ Use FTOC.



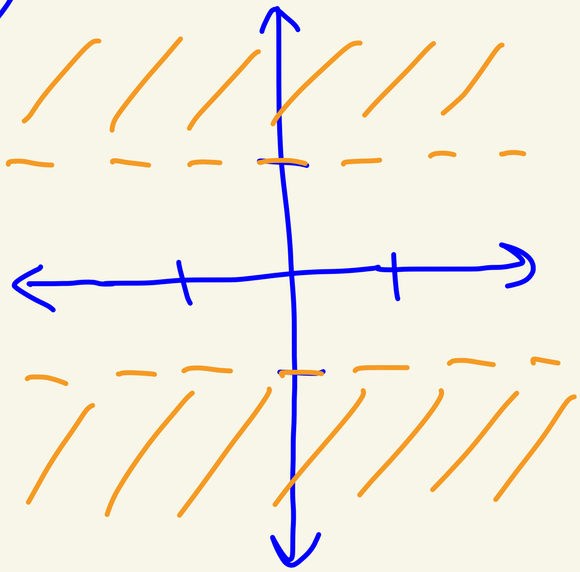
$$\int_\gamma \left(2\sin(z)+z^3\right)dz$$

$$= -2\cos(z) + \frac{z^4}{4}\Big|_{1}^{-1}$$

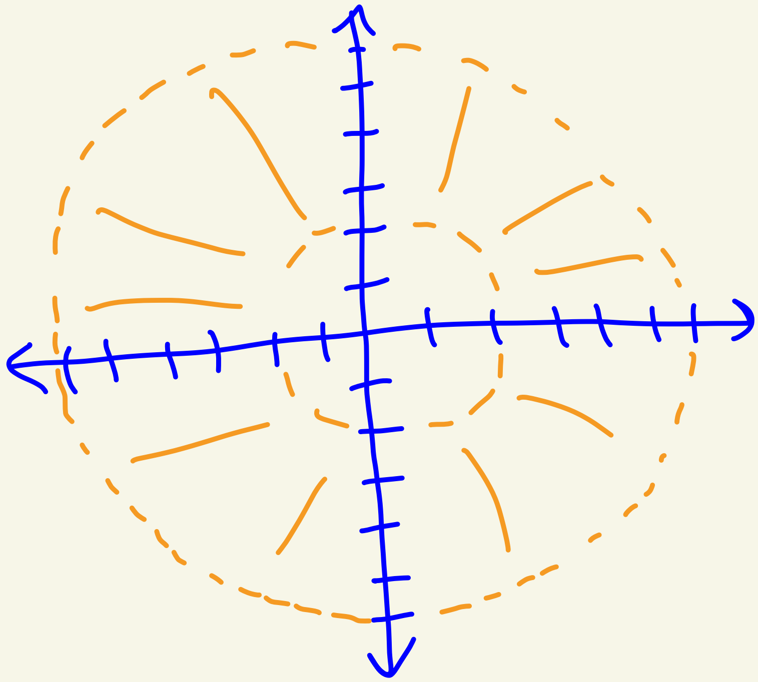
$$= \underbrace{-2\cos(-1)}_{2\cos(1)} + \frac{(-1)^4}{4} - \left[-2\cos(1) + \frac{(1)^4}{4}\right]$$

$$= -2\cos(1) + \frac{1}{4} + 2\cos(1) - \frac{1}{4} = \boxed{0}$$

④ $|\mathrm{Im}(z)| > 1$

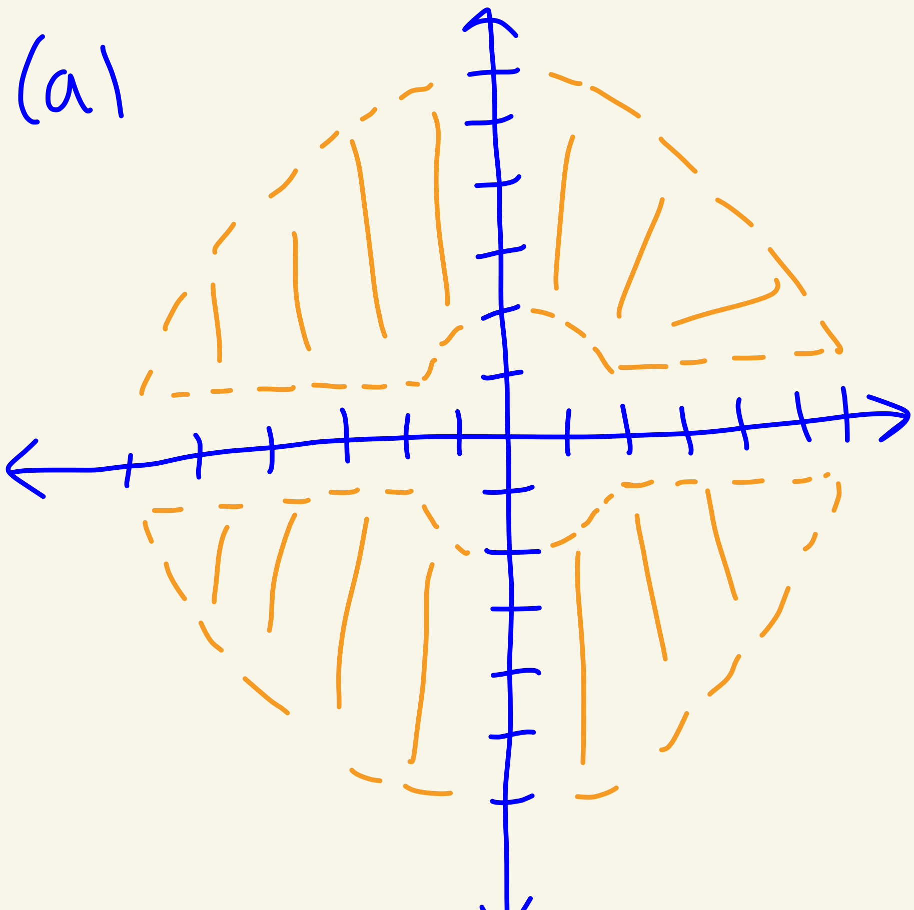


$\mathrm{Im}(z) < -1$ or $\mathrm{Im}(z) > 1$



$2 < |z| < 6$

(a)



(b) S is not path connected.

S is open.

(c) S is not a region.

A or B

A This is HW 4 problem 6.

B This is HW 4 problem 4.