Hamiltonicity and Circular Distance Two Labellings *

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Abstract

A k-circular distance two labelling (or k-c-labelling) of a graph G is a vertex-labelling such that the circular difference (mod k) of the labels is at least two for adjacent vertices, and at least one for vertices at distance two. Given G, denote $\sigma(G)$ the minimum k for which there exists a k-c-labelling of G. Suppose G has n vertices, we prove $\sigma(G) \leq n$ if G^c is Hamiltonian; and $\sigma(G) = n + p_v(G^c)$ otherwise, where $p_v(G)$ is the path covering number of G. We give exact values of $\sigma(G)$ for some families of graphs such that G^c is Hamiltonian, and discuss injective k-c-labellings especially for joins and unions of graphs.

Keywords. Hamiltonicity, vertex-labelling, $L(2,1)$-labelling, path covering number.

1 Introduction

Motivated from the channel assignment problem introduced by Hale [5], the distance two labelling was first introduced and studied by Griggs and Yeh [4]. Given a graph G, for any $u, v \in V(G)$, let $\mathrm{d}_G(u,v)$ denote the distance between u and v in G. An $L(2,1)$-*labelling* is a function $f : V(G) \to \{0, 1, 2, \cdots\}$ such that if $uv \in E(G)$ then $|f(u) - f(v)| \geq 2$; and if $\mathrm{d}_G(u,v) = 2$, then $|f(u) - f(v)| \geq 1$. The *span* of an $L(2,1)$-labelling f is defined as $\max\limits_{u,v \in V(G)} |f(u) - f(v)|$. The λ-*number*, $\lambda(G)$, is the minimum span among all $L(2,1)$-labellings of G.

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We consider a variation of the $L(2,1)$-labelling by using a different measurement. For a positive integer k, a *k-circular-labelling* (or k-c-labelling for short) of a graph G is a function, $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$, such that:

$$|f(u) - f(v)|_k \geq \begin{cases} 2, & \text{if } \mathrm{d}_G(u,v) = 1; \\ 1, & \text{if } \mathrm{d}_G(u,v) = 2. \end{cases}$$

where $|x|_k := \min\{|x|, k - |x|\}$ is the *circular difference* modulo k. The *σ-number*, $\sigma(G)$, is the minimum k of a k-c-labelling of G. A generalization of this labelling, namely, circular distance d labelling (with restrictions on vertices of distance $\leq d$), was introduced and studied by ven den Heuvel, Leese and Shepherd [6].

In this Note, only finite simple graphs are considered. To find the minimum span, we consider without loss of generality only the labellings in which 0 is used. Given a graph G, the *path covering number*, $p_v(G)$, is the smallest number of vertex-disjoint paths covering $V(G)$. Georges, Mauro and Whittlesey [3] proved the following result:

Theorem 1.1 *[3] Given a graph G on n vertices, then*

$$\lambda(G) \begin{cases} \leq n - 1, & \text{if } p_v(G^c) = 1; \\ = n + p_v(G^c) - 2, & \text{if } p_v(G^c) \geq 2. \end{cases}$$

It is known [6] and not hard to observe the following inequalities:

$$\lambda(G) + 1 \leq \sigma(G) \leq \lambda(G) + 2, \quad \text{for any graph } G. \qquad (*)$$

In this Note, we use Theorem 1.1 and (*) to prove:

Theorem 1.2 *Given a graph G on n vertices, then*

$$\sigma(G) \begin{cases} \leq n, & \text{if } G^c \text{ is Hamiltonian;} \\ = n + p_v(G^c), & \text{if } G^c \text{ is not Hamiltonian.} \end{cases}$$

In Section 3, we give sufficient conditions for each of the two inequalities in (*), and determine the σ-numbers for cycles and trees. In Section 4, we study injective circular distance two labellings, especially for unions and joins of graphs.

2 Proof of Theorem 1.2

If L is a k-c-labelling of a graph G, define the following for $0 \le i \le k-1$:

$L_i := \{v : L(v) = i\}$ and $l_i := |L_i|$;

$H(L) := \{i : L_i = \emptyset\}$;

$G(L) := \{i : L_i = \emptyset$ and $l_{i-1} = l_{i+1} = 1\}$;

$M(L) := \{i : l_i \ge 2\}$.

All the indices above are taken (mod k). If $i \in H(L), G(L)$ or $M(L)$, then i is called a *hole, gap* or *multiplicity* of L, respectively. Given G, a k-c-labelling is a *σ-labelling* if $k = \sigma(G)$. A σ-labelling is *min-hole* if it has the minimum number of holes among all σ-labellings of G.

Theorem 2.1 *If G has n vertices and $\sigma(G) \ge n+1$, then $\sigma(G) = \lambda(G) + 2$.*

Proof. Suppose $\sigma(G) \ge n+1$. Let L be a σ-labelling, then $H(L) \ne \emptyset$. Without loss of generality, assume $L_{\sigma-1} = \emptyset$. Since L is also an $L(2,1)$-labelling, so $\lambda(G) \le \sigma(G) - 2$. By (*), $\sigma(G) = \lambda(G) + 2$. Q.E.D.

By Theorems 1.1 and 2.1, to prove Theorem 1.2 it remains to show that G^c is Hamiltonian if and only if $\sigma(G) \le n$. Thus it suffices to prove the following:

Theorem 2.2 *Let G be a graph on n vertices. Suppose L is a min-hole σ-labelling of G, the following are equivalent:*

(1) $G(L) = \emptyset$;

(2) G^c is Hamiltonian;

(3) $\sigma(G) \le n$.

We shall prove Theorem 2.2 by using the next three lemmas.

Lemma 2.3 *Let L be a min-hole σ-labelling of G. If $h \in H(L)$, then $l_{h-1} = l_{h+1} > 0$, and the subgraph of G induced by $L_{h-1} \cup L_{h+1}$ is a perfect matching, where the indices are taken modulo $\sigma(G)$.*

Proof. Let $\sigma(G) = k$. Suppose $h \in H(L)$, i.e., $L_h = \emptyset$. Since L is a σ-labelling, it is impossible to have two consecutive holes. Hence $l_{h-1}, l_{h+1} > 0$.

Observe that each vertex in L_{h-1} is adjacent to at most one vertex in L_{h+1}, and vice versa. It suffices to show that each vertex in L_{h-1} is adjacent to L_{h+1} (it is symmetrical to show that each vertex in L_{h+1} is adjacent to L_{h-1}). Suppose to the contrary, there exists $v \in L_{h-1}$ such that v is not adjacent to L_{h+1}. Without loss of generality, assume $h - 1 = 0$. There are two cases.

Case 1: If $L_0 = \{v\}$. Define a function L' on $V(G)$ by $L'(u) = L(u) - 1$ if $u \neq v$; $L'(v) = L(v) = 0$. By the assumption that v is not adjacent to L_{h+1}, one can verify that L' is a $(k-1)$-c-labelling of G, a contradiction.

Case 2: If $\{u, v\} \subseteq L_0$. Define a function L' on $V(G)$ by $L'(x) = L(x)$ if $x \neq v$; $L'(v) = 1$. Then L' is a σ-labelling with fewer holes than L, a contradiction. Q.E.D.

Lemma 2.4 *If L is a min-hole σ-labelling of G, then $G(L) = \emptyset$ or $M(L) = \emptyset$.*

Proof. Let $\sigma(G) = k$. Suppose L is a min-hole σ-labelling of G with $G(L) \neq \emptyset$ and $M(L) \neq \emptyset$. Let $g \in G(L)$ and $m \in M(L)$ such that $|g - m|_k$ is the smallest. Without loss of generality, assume $m = 0$ and $g < k/2$. Then $g \geq 2$ and $l_i = 1$ for all $i = 1, 2, \cdots g - 1, g + 1$.

Let $L_{g-1} = \{v_{g-1}\}$, $L_{g+1} = \{v_{g+1}\}$, then any vertex in L_0 is adjacent to v_{g-1} or v_{g+1}. For otherwise, if there exists $v \in L_0$ with $vv_{g-1}, vv_{g+1} \notin E(G)$, then defining $L'(v) = g$ and $L'(u) = L(u)$ for $u \neq v$ results in a k-c-labelling with fewer holes. Since both v_{g-1} and v_{g+1} are adjacent to at most one vertex in L_0, we conclude that $l_0 = 2$. Let $L_0 = \{x, y\}$ so that xv_{g-1}, $yv_{g+1} \in E(G)$, and xv_{g+1}, $yv_{g-1} \notin E(G)$. Define:

$$L'(v) = \begin{cases} g - L(v), & \text{if } 1 \leq L(v) \leq g - 1; \\ g, & \text{if } v = x; \\ L(v), & \text{otherwise.} \end{cases}$$

One can verify that L' is a σ-labelling with fewer holes, a contradiction. Q.E.D.

Suppose f is a k-c-labelling of G. For any $u, v \in V(G)$, if $f(u) = f(v)$ or $f(u) \equiv f(v) \pm 1 \pmod{k}$, then $uv \in E(G^c)$. The following lemma can be proved easily.

Lemma 2.5 *If f is a k-c-labelling of G with $H(f) = \emptyset$, then G^c is Hamiltonian.*

Proof of Theorem 2.2. (1) $\Rightarrow$ (2): By Lemma 2.5, it suffices to consider that $H(L) \neq \emptyset$. Let $h \in H(L)$, since $G(L) = \emptyset$, by Lemma 2.3, we have $l_{h-1} = l_{h+1} \geq 2$ and there exist $v_{h-1} \in L_{h-1}$, $v_{h+1} \in L_{h+1}$ such that $v_{h-1}v_{h+1} \in E(G^c)$.

To get a Hamilton cycle in G^c, first trace the vertices in L_0, L_1, L_2, $\cdots$ successively until there is a hole h. From the previous paragraph, there exists $v_{h-1}v_{h+1} \in E(G^c)$. Hence, the process can be continued until a Hamilton cycle is obtained.

(2) $\Rightarrow$ (3): Suppose G^c has a Hamilton cycle, $v_0, v_1, \cdots, v_{n-1}, v_0$, then the labelling $L(v_x) = x$ is an n-c-labelling of G. Hence $\sigma(G) \leq n$.

(3) $\Rightarrow$ (1): Suppose $\sigma(G) \leq n$. Let L be a min-hole σ-labelling of G. If $G(L) \neq \emptyset$, by Lemma 2.5, $M(L) = \emptyset$. Hence L is injective, which is impossible since $\sigma \leq n$ and $G(L) \neq \emptyset$. Q.E.D.

The following corollary follows immediately from Theorems 1.1, 1.2 and 2.1.

Corollary 2.6 *If G is a graph on n vertices, the following are equivalent:*

(1) $\sigma(G) = n + 1$;

(2) $\sigma(G) = n + 1$ *and* $\lambda(G) = n - 1$;

(3) $p_v(G^c) = 1$, *and G^c is not Hamiltonian.*

Denote the union of two vertex-disjoint graphs G and H by $G \cup H$. The *join* of G and H is the graph $G \vee H$ obtained from $G \cup H$ by joining each vertex in G to each vertex in H. For any integers p and q with $p < q/2$, define the graph $G_{p,q} = K_p \vee (K_p^c \cup K_{q-2p})$, where K_n is a complete graph on n vertices. Chvátal [2] proved that $G_{p,q}$ is maximal non-Hamiltonian. Thus by Corollary 2.6, we have:

Corollary 2.7 *If $G = G_{p,q}^c$, then $\sigma(G) = q + 1$ and $\lambda(G) = q - 1$.*

3 Graphs with Hamiltonian Complements

For any G, by (*), $\sigma(G)$ is either $\lambda(G) + 1$ or $\lambda(G) + 2$. If G^c is not Hamiltonian, by Theorems 1.2 and 2.1, $\sigma(G) = \lambda(G) + 2$. We show both possible values of $\sigma(G)$,

$\lambda(G)+1$ and $\lambda(G)+2$, are attained by some graphs with Hamiltonian complements.

We start with diameter two graphs for which any distance two labelling is one-to-one. By Theorems 1.1 and 1.2, we have:

Theorem 3.1 *Suppose G is a graph on n vertices. If G is of diameter two and G^c is Hamiltonian, then $\sigma(G) = \lambda(G) + 1 = n$.*

An example of Theorem 3.1 is the Petersen Graph. Another example is the Cartesian product of complete graphs $K_m \times K_n$, $m, n > 2$. The *Cartesian product of graphs* G and H, $G \times H$, has the vertex set $V = \{(u, v) : u \in G, v \in H\}$ and edge set $E = \{(u, v)(w, x) : (u = w \text{ and } vx \in E(H)) \text{ or } (v = x \text{ and } uw \in E(G))\}$.

Theorem 3.2 *For any $m, n \geq 2$, let $G = K_m \times K_n$, then*

$$\sigma(G) = \begin{cases} \lambda(G) + 2 = 6, & \text{if } m = n = 2; \\ \lambda(G) + 1 = mn, & \text{otherwise.} \end{cases}$$

Proof. If $m = n = 2$, then $p_v(G^c) = 2$, so $\sigma(G) = 6$.

Suppose $m \leq n$. Since G has diameter two and mn vertices, by Theorem 3.1, it suffices to show that G^c is Hamiltonian. Let $V(K_m) = \{u_1, u_2, \cdots, u_m\}$ and $V(K_n) = \{v_1, v_2, \cdots, v_n\}$, then $E(G^c) = \{(u_i v_j)(u_k v_l) : i \neq k, j \neq l\}$. For the two cases: ($m = 2$ and $n \geq 3$) and ($m = n = 3$), one can find the Hamilton cycles in G^c, respectively: $(u_2v_2), (u_1v_1), (u_2v_3), (u_1v_2), (u_2v_4), (u_1v_3), \cdots, (u_2v_n), (u_1v_{n-1}), (u_2v_1), (u_1v_n), (u_2v_2)$; and $(u_1v_1), (u_2v_2), (u_3v_1), (u_1v_2), (u_2v_3), (u_3v_2), (u_1v_3), (u_2v_1), (u_3v_3), (u_1v_1)$.

If $m \geq 3$, $n \geq 4$, then $(K_m \times K_n)^c$ is regular with degree $(m-1)(n-1) \geq mn/2$. By the well-known Dirac Theorem, G^c is Hamiltonian. Q.E.D.

The result for $\lambda(K_m \times K_n)$ in Theorem 3.2 was proved by Georges et al [3].

Now we focus on cycles and trees. For any cycle, Griggs and Yeh [4] proved that the λ-number is 4. However, the σ-number has two possible values.

Theorem 3.3 *For the cycle C_n on n vertices, $n \geq 3$,*

$$\sigma(C_n) = \begin{cases} 5, & \text{if } n \equiv 0 \pmod 5; \\ 6, & \text{if } n \not\equiv 0 \pmod 5. \end{cases}$$

Proof. Since $\lambda(C_n) = 4$ [4], by (*), $5 \le \sigma(C_n) \le 6$. Suppose $\sigma(C_n) = 5$. Let f be a 5-c-labelling of C_n, $f(V) \subseteq \{0, 1, 2, 3, 4\}$. Assume $f(v) = 0$ for some v, then the labels for the two neighbors of v must be 2 and 3. Indeed, if $f(u) = x$, then the labels for the two neighbors of u must be $x + 2$ and $x + 3 \pmod 5$. This implies that the labelling is well-defined only when $n \equiv 0 \pmod 5$. Q.E.D.

Let T be a tree with maximum degree Δ. Griggs and Yeh [4] proved that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$. Chang and Kuo [1] gave a polynomial algorithm determining the λ-number for trees.

If T is a tree with maximum degree Δ, then clearly $\sigma(T) \ge \Delta + 3$. Furthermore, a $(\Delta + 3)$-c-labelling for T can be obtained by using a greedy (first-fit) algorithm starting with a vertex of degree Δ. Thus, we have

Theorem 3.4 *If T is a tree with maximum degree Δ, then $\sigma(T) = \Delta + 3$.*

4 Injective Distance Two Labellings

A one-to-one k-c-labelling (or $L(2, 1)$-labelling, respectively) is called a *k-c′-labelling* (or *$L'(2, 1)$-labelling*, respectively). The parameter $\sigma'(G)$ is the minimum k for which a k-c′-labelling exists, and $\lambda'(G)$ is the minimum span of an $L'(2, 1)$-labelling.

The following result was proved, independently, by Georges et al. [3], and by Chang and Kuo [1].

Theorem 4.1 *[3, 1] If G is a graph on n vertices, then $\lambda'(G) = n + p_v(G^c) - 2$.*

Theorem 4.2 *If G is a graph on n vertices, then*

$$\sigma'(G) = \begin{cases} n, & \text{if } G^c \text{ is Hamiltonian;} \\ n + p_v(G^c), & \text{otherwise.} \end{cases}$$

Equivalently, $\sigma'(G) = \max\{n, \sigma(G)\}$.

Proof. Clearly $\sigma'(G) \ge \max\{\sigma(G), n\}$. If G^c has a Hamilton cycle, $v_0, v_1, \cdots, v_{n-1}, v_0$, then $L(v_i) = i$, $0 \le i \le n - 1$, is an n-c′-labelling, so $\sigma'(G) = n$. If G^c is not Hamiltonian, let L be a min-hole σ-labelling. By Theorem 2.2 and Lemma 2.4, L is injective. Thus $\sigma'(G) = \sigma(G)$. Q.E.D.

For joins and unions of graphs G and H, observe that $(G \vee H)^c = G^c \cup H^c$ and $(G \cup H)^c = G^c \vee H^c$. Moreover, it is easy to learn that $p_v(G \cup H) = p_v(G) + p_v(H)$, so $p_v((G \vee H)^c) = p_v(G^c) + p_v(H^c) \geq 2$. The following result follows immediately from Theorems 1.2, 4.1 and 4.2:

Theorem 4.3 *Given m graphs $G_1, G_2, \cdots, G_m$, let $G = G_1 \vee G_2 \cdots \vee G_m$. Then $\sigma'(G) = \sigma(G) = \sum_{i=1}^{m} \{\lambda'(G_i) + 2\}$.*

The *wheel* with n spokes, W_n, $n \geq 3$, is the join of the cycle C_n with a single vertex, i.e., $W_n = C_n \vee \{v\}$. By Theorems 3.3 and 4.3, $\sigma'(W_n) = \sigma(W_n) = 8$, if $n = 3, 4$; and $\sigma'(W_n) = \sigma(W_n) = n + 3$, if $n > 4$.

To find the σ'-number for unions of graphs, we make use of the following result of Chang and Kuo [1].

Theorem 4.4 *[1] For any G and H, $p_v(G \vee H) = \max\{p_v(G) - |V(H)|, p_v(H) - |V(G)|, 1\}$.*

Theorem 4.5 *If G and H are graphs on m and n vertices respectively, then $\sigma'(G \cup H) = \max\{\sigma'(G), \sigma'(H), m + n\}$.*

Proof. It is obvious that $\sigma'(G \cup H) \geq \max\{\sigma'(G), \sigma'(H), m + n\}$. If $(G \cup H)^c$ is Hamiltonian, then by Theorem 4.2, $\sigma'(G \cup H) = m + n \leq \max\{\sigma'(G), \sigma'(H), m + n\}$.

If $(G \cup H)^c = G^c \vee H^c$ is not Hamiltonian, then $p_v(G^c) > n$ or $p_v(H^c) > m$ (for if $p_v(G^c) \leq n$ and $p_v(H^c) \leq m$, then $G^c \vee H^c$ is Hamiltonian). By Theorem 4.4, without loss of generality, assume $p_v(G^c \vee H^c) = p_v(G^c) - n \geq 1$. Since $G^c \vee H^c$ is not Hamiltonian, by Theorem 4.2, $\sigma'(G \cup H) = m + n + p_v(G^c \vee H^c) = m + n + p_v(G^c) - n = m + p_v(G^c) = \sigma'(G)$ (since $p_v(G^c) \geq 2$) $\leq \max\{\sigma'(G), \sigma'(H), m + n\}$. Q.E.D.

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