Notes on Dynamics

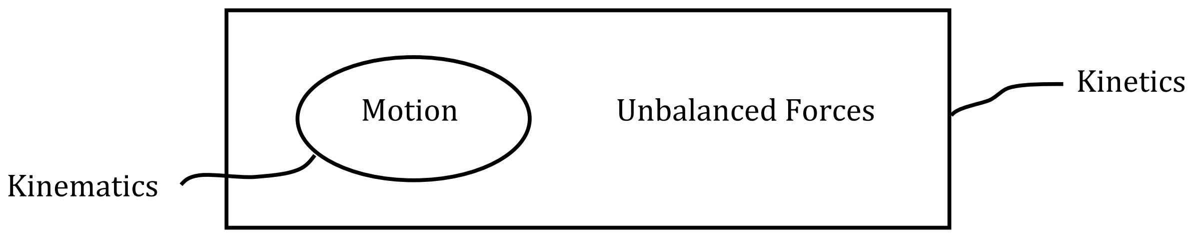
by

Stephen F. Felszeghy
CSULA Prof. Emeritus of ME

These notes are a supplement to *FE Reference Handbook*, 10.2 Edition, for the Computer-Based Exam, NCEES, July 2022, pp. 114-129.

These notes were prepared for the FE/EIT Exam Review Course class meeting held on Oct. 15, 2022, 9:00 a.m. to 12:00 p.m.

Dynamics

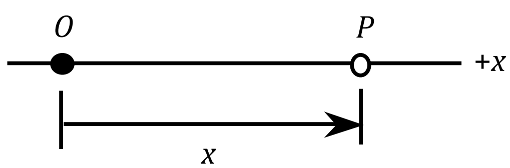


Kinematics – deals with motion alone apart from considerations of force and mass.

Kinetics – relates unbalanced forces with changes in motion.

Kinematics of Particles

Rectilinear Motion of a Particle



Position coordinate
(Rectilinear displacement): $x = f(t) \rightarrow x = x(t)$

Velocity: $v = \dfrac{dx}{dt} = \dot{x}$

Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2} = \ddot{x}$

Suppose $v = v(x)$; apply "Chain Rule":

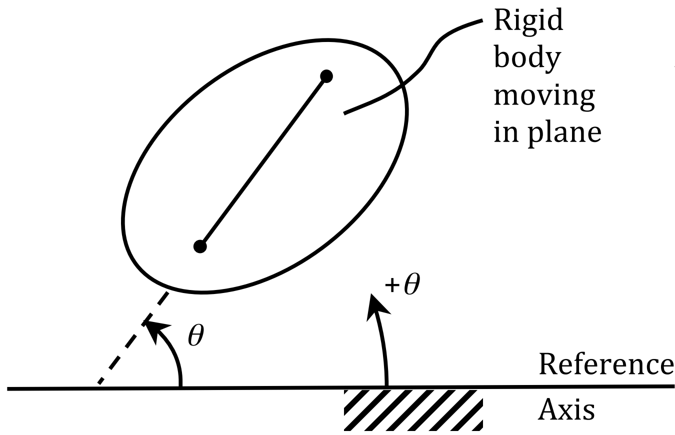
$$\frac{dv}{dt} = a = \frac{dv}{dx}\frac{dx}{dt} \quad \rightarrow \quad a = \frac{dv}{dx}v$$

Determination of Motion of a Particle

Integrate differential relations:

$$dx = v\,dt$$
$$dv = a\,dt$$
$$v\,dv = a\,dx$$

Angular Motion of a Line



Angular position coordinate
(Angular displacement): $\theta = f(t)$

Angular velocity: $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$

Angular acceleration: $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2} = \ddot{\theta}$

Differential relations:

$$d\theta = \omega\, dt$$
$$d\omega = \alpha\, dt$$
$$\omega\, d\omega = \alpha\, d\theta$$

Note analogy with rectilinear motion.

Two common cases:

1. Acceleration a = constant, or α = constant

2. Acceleration $a = f(t)$, or $\alpha = f(t)$

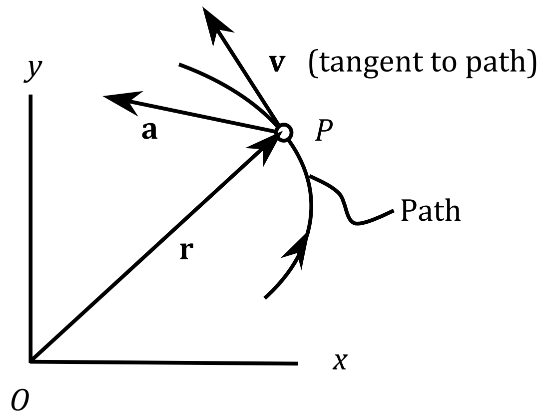
See motion equations in the 10.2 Handbook on pp. 117-118.

Curvilinear Motion of a Particle

Vectors will be denoted by upright boldface letters, e.g., **r**.

Vectors will be denoted by underlined letters in handwriting, e.g., <u>r</u>.

Scalar component of vector **r** will be denoted by *italic r*.

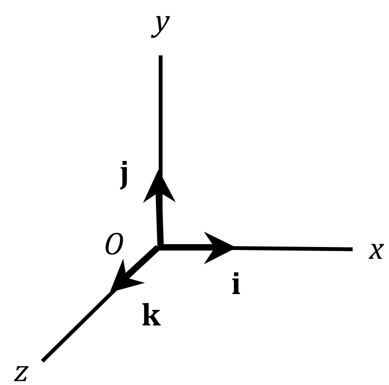


Position vector: $\mathbf{r} = \mathbf{r}(t)$
(Vector function)

Velocity: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$

Acceleration: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}$

<u>Rectangular Components</u>



Position vector: $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$

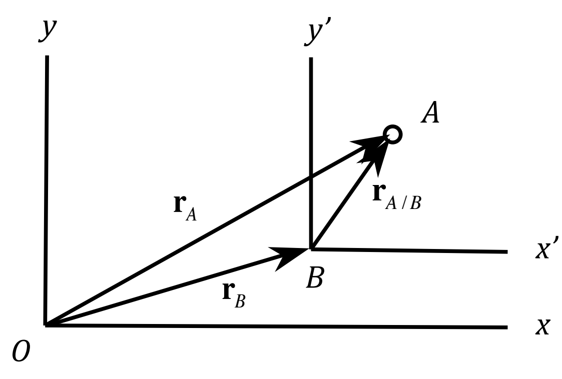
Velocity: $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\mathbf{k}$

Acceleration: $\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\mathbf{k}$

We write: $v_x = \dot{x}$, etc.
$a_x = \ddot{x}$, etc.

Application: See projectile motion in the 10.2 Handbook on p. 118.

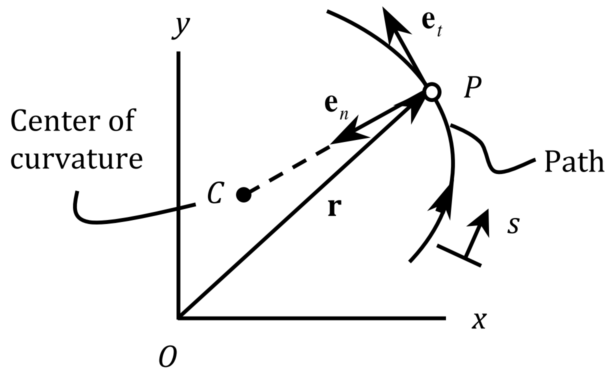
<u>Motion Relative to Translating Reference Axes</u>



"Translating" means *x' – y'* axes move but remain parallel to *x – y* axes.

$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$
$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_{A/B}$
$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$
$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$

Tangential and Normal Components



$|CP| = \rho =$ radius of curvature

$\mathbf{e}_t$ = unit vector tangent to path

$\mathbf{e}_n$ = unit vector normal to path pointing to C

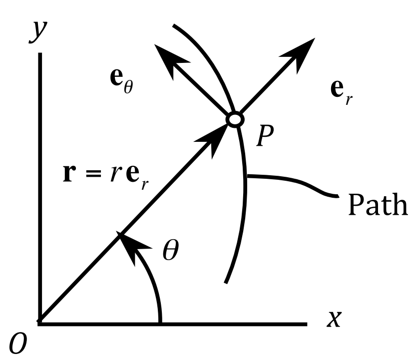
s = directed distance along path

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\mathbf{e}_t = v\,\mathbf{e}_t$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2 s}{dt^2}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$

$$= a_t\mathbf{e}_t + a_n\mathbf{e}_n = \mathbf{a}_t + \mathbf{a}_n$$

Radial and Transverse Components



Polar coordinates of P: $(r,\ \theta)$

$\mathbf{e}_r$ = unit vector in $\mathbf{r}$ direction

$\mathbf{e}_\theta$ = unit vector perpendicular to $\mathbf{r}$ in direction of increasing θ

$$\mathbf{r} = r\,\mathbf{e}_r$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

If path is a circle, then r = constant, $\dot{r} = \ddot{r} = 0$,

$$\mathbf{v} = r\dot{\theta}\mathbf{e}_\theta$$
$$\mathbf{a} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta$$

Kinetics of Particles: Newton's Second Law

$$\sum \mathbf{F} = m\mathbf{a} \qquad \text{(Equation of motion)}$$

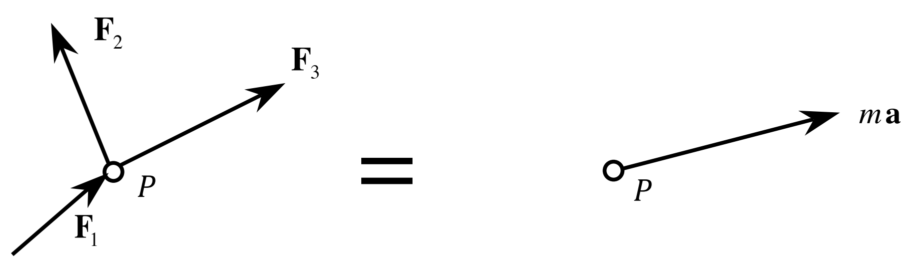
where

$\sum \mathbf{F}$ = resultant force

m = mass of particle

$\mathbf{a}$ = absolute acceleration, measured in a newtonian frame of reference (inertial system)

Graphical Representation of Newton's 2nd Law



Free-body diagram (FBD)

Kinetic diagram (KD) (Mass-acceleration diagram)

Units

Quantity / System	Length	Time	Mass	Force
SI	m	s	kg	$\text{N} = \text{kg} \cdot \frac{\text{m}}{\text{s}^2}$
USCS	ft	s	$\text{slug} = \text{lb} \cdot \frac{\text{s}^2}{\text{ft}}$	lb

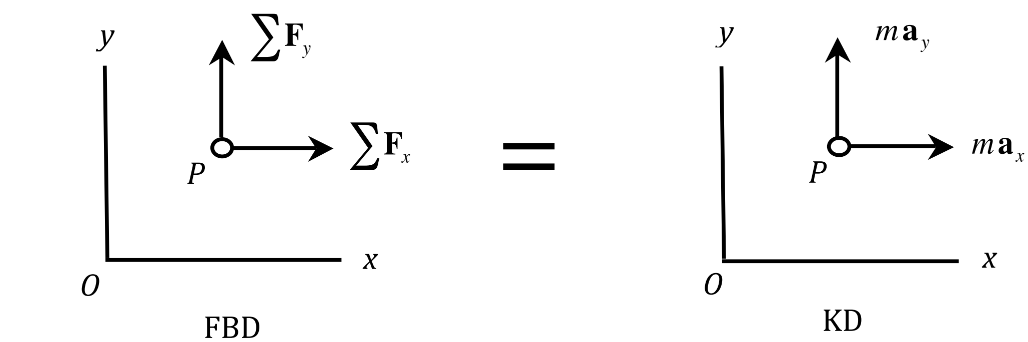
In either system, $W = mg$, where

W = weight
g = acceleration due to gravity

At surface of earth: (SI) $g = 9.807\ \text{m/s}^2$
(USCS) $g = 32.174\ \text{ft/s}^2$

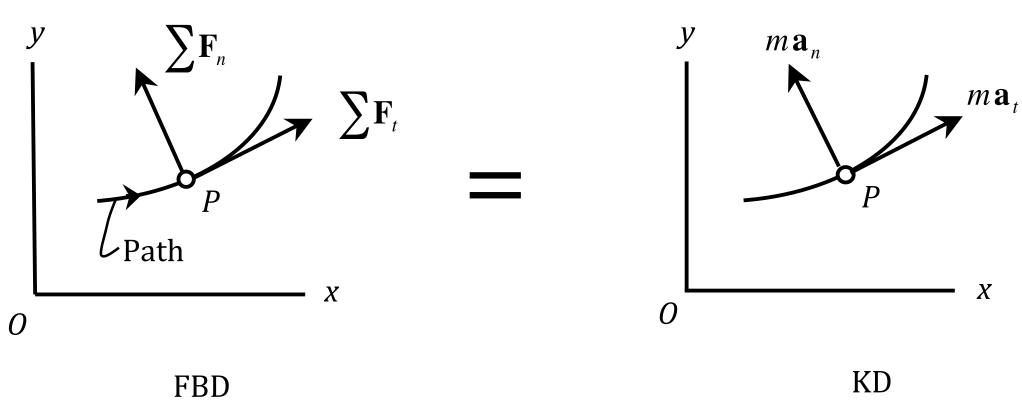
AVOID: lbf, lbm

Equations of Motion: Rectangular Components



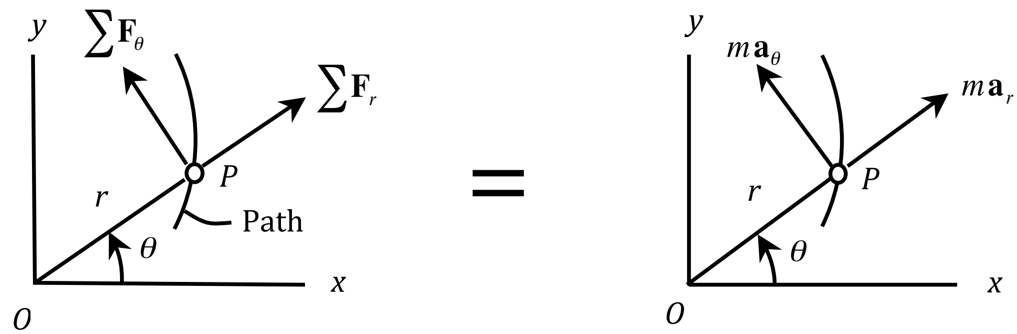
$$\sum F_x = m a_x$$
$$\sum F_y = m a_y$$

Equations of Motion: Tangential and Normal Components



$$\sum F_t = m a_t$$
$$\sum F_n = m a_n$$

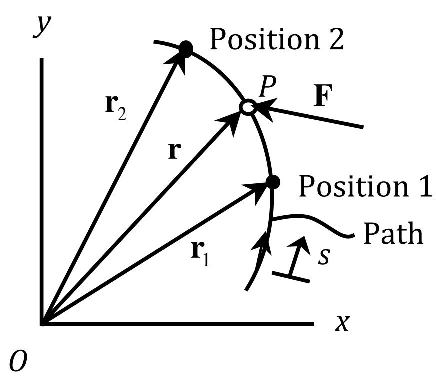
Equations of Motion: Radial and Transverse Components



FBD KD

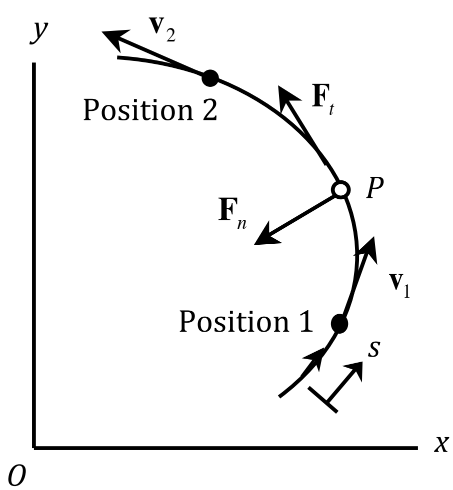
$$\sum F_r = m a_r$$
$$\sum F_\theta = m a_\theta$$

Kinetics of Particles: Energy Methods



The *work* done by **F** on the particle during a finite movement of the particle along a curved path from position 1 to position 2 is $U_{1\to 2}$:

$$U_{1\to 2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \qquad \text{(Line integral)}$$



It can be shown:

$$U_{1\to 2} = \int_{s_1}^{s_2} F_t \, ds$$
$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

Let $T = \frac{1}{2} m v^2 =$ *kinetic energy* of particle

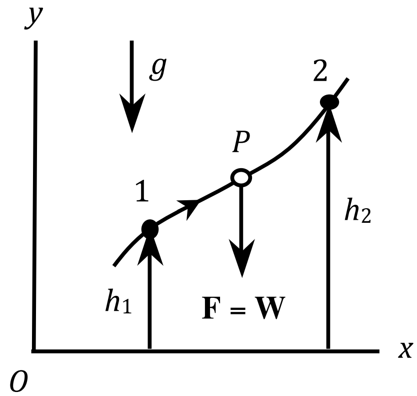
Then,

$$U_{1\to 2} = T_2 - T_1$$
$$= \Delta T$$

or $\quad T_2 = T_1 + U_{1\to 2}$

Above result is the *principle of work and energy*.   Units: (SI) N·m = J; (USCS) ft·lb

Work Done on Particle by Gravitational Force



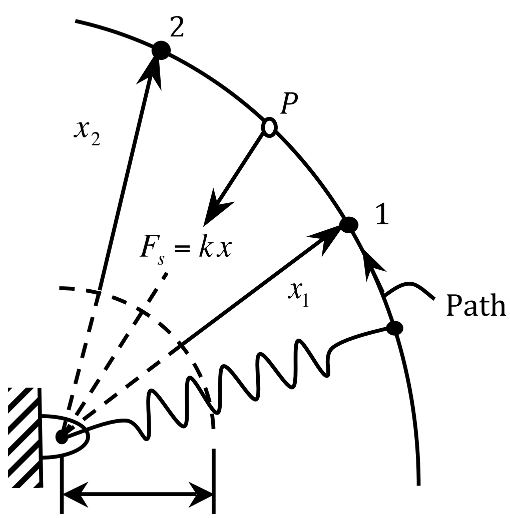
$$U_{1\to 2} = -\int_{h_1}^{h_2} W\,dy = -(Wh_2 - Wh_1)$$

Let $V_g = Wy = mgy =$ *gravitational potential energy* of particle

Then, $U_{1\to 2} = -\left[(V_g)_2 - (V_g)_1\right]$
$= -\Delta V_g$

Note $U_{1\to 2}$ is independent of path from 1 to 2. For this reason **W** is called a *conservative force*.

Work Done on Particle by a Linearly-Elastic Spring Force



Let k = spring constant
x = spring elongation
$F_s = kx =$ spring force

Then,

$$U_{1\to 2} = -\int_{x_1}^{x_2} F_s\,dx = -\int_{x_1}^{x_2} kx\,dx$$
$$= -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right)$$

Let $V_e = \frac{1}{2}kx^2 =$ *elastic potential energy* of particle

Then, $U_{1\to 2} = -\left[(V_e)_2 - (V_e)_1\right]$
$= -\Delta V_e$

Note $U_{1\to 2}$ is independent of path from 1 to 2. For this reason F_s is called a *conservative force.*

Summary

The work-energy equation can now be written as:

$$U_{1\to 2} = \Delta T + \Delta V_g + \Delta V_e$$

where $U_{1\to 2}$ is the work done on the particle by forces other than gravitational and spring forces.

If $U_{1\to 2}$ above is zero, then:

$$T_2 + \left(V_g\right)_2 + \left(V_e\right)_2 = T_1 + \left(V_g\right)_1 + \left(V_e\right)_1$$

This is the *law of conservation of total mechanical energy*.

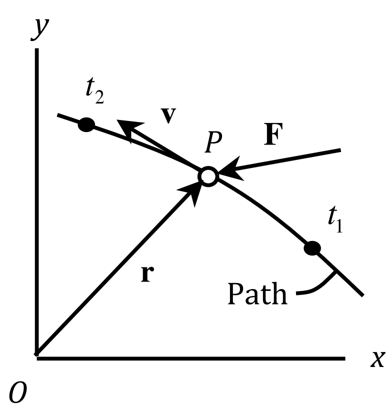
Power and Efficiency

Power is the time rate of doing work by a force on a particle.

$$\text{Power} = \mathbf{F}\cdot\mathbf{v} \qquad \text{Units: } (\text{SI})\ \text{N}\cdot\text{m/s} = \text{J/s} = \text{W};\ (\text{USCS})\ \text{hp} = 550\ \frac{\text{ft}\cdot\text{lb}}{\text{s}}$$

$$\eta = \frac{\text{power output}}{\text{power input}} = \text{ mechanical efficiency}$$

Kinetics of Particles: Momentum Methods



Recall Newton's 2nd law:

$$\mathbf{F} = m\mathbf{a} = \frac{d}{dt}\left(m\mathbf{v}\right)$$

where $\mathbf{F} =$ resultant force

$m\mathbf{v} =$ *linear momentum* of particle

Define *angular momentum* $\mathbf{H}_O$ of particle about O:

$$\mathbf{H}_O = \mathbf{r}\times m\mathbf{v}$$

$$\text{Then, } \dot{\mathbf{H}}_O = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \mathbf{F} = \mathbf{M}_O$$

$$\text{or } \mathbf{M}_O = \dot{\mathbf{H}}_O$$

where $\mathbf{M}_O =$ sum of the moments about O of all forces acting on particle

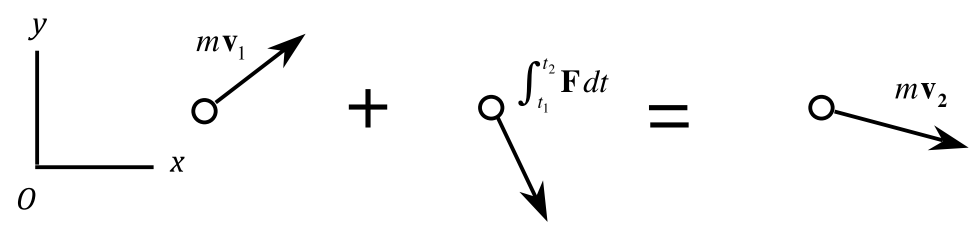
<u>Equations of Impulse and Momentum</u>

What is the cumulative effect of integrating $\mathbf{F}$ and $\mathbf{M}_O$ with respect to time over an interval from t_1 to t_2?

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = \int_{m\mathbf{v}_1}^{m\mathbf{v}_2} d(m\mathbf{v}) = m\mathbf{v}_2 - m\mathbf{v}_1$$

$$\text{or} \quad m\mathbf{v}_1 + \int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2$$

Graphical interpretation:



Initial linear momentum | *Linear impulse* | Final linear momentum

$$\text{or} \quad (mv_x)_1 + \int_{t_1}^{t_2} F_x\,dt = (mv_x)_2$$
$$(mv_y)_1 + \int_{t_1}^{t_2} F_y\,dt = (mv_y)_2$$

Units: $(\text{SI})\ \text{kg} \cdot \frac{\text{m}}{\text{s}} = \text{N} \cdot \text{s}; \ (\text{USCS})\ \text{lb} \cdot \text{s}$

Recall $\mathbf{M}_O = \frac{d\mathbf{H}_O}{dt}$

$$\int_{t_1}^{t_2} \mathbf{M}_O\,dt = \int_{(\mathbf{H}_O)_1}^{(\mathbf{H}_O)_2} d\mathbf{H}_O = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

or $\left(\mathbf{H}_O\right)_1 + \int_{t_1}^{t_2} \mathbf{M}_O \, dt = \left(\mathbf{H}_O\right)_2$

Initial angular momentum	+	*Angular impulse*	=	Final angular momentum

Units: $(\text{SI})\ \text{kg} \cdot \dfrac{\text{m}^2}{\text{s}} = \text{N} \cdot \text{m} \cdot \text{s};\ (\text{USCS})\ \text{lb} \cdot \text{ft} \cdot \text{s}$

<u>Extension to System of n Particles</u>

Let $\sum m\mathbf{v} = \sum_{i=1}^{n} m_i \mathbf{v}_i$

$$\sum m\mathbf{v}_1 \quad + \quad \sum \int_{t_1}^{t_2} \mathbf{F} \, dt \quad = \quad \sum m\mathbf{v}_2$$

Sum of all initial linear momenta	Sum of all linear impulses from *external* forces	Sum of all final linear momenta

Note: Linear impulses from *internal* forces of action and reaction cancel.

If no external forces act from time t_1 to t_2, then

$$\sum m\mathbf{v}_1 = \sum m\mathbf{v}_2$$

and the *total linear momentum* of the particles is *conserved*.

Let $\sum \mathbf{H}_O = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{v}_i$

$$\sum \left(\mathbf{H}_O\right)_1 \quad + \quad \sum \int_{t_1}^{t_2} \mathbf{M}_O \, dt \quad = \quad \sum \left(\mathbf{H}_O\right)_2$$

Sum of all initial angular momenta	Sum of all angular impulses from *external* forces	Sum of all final angular momenta

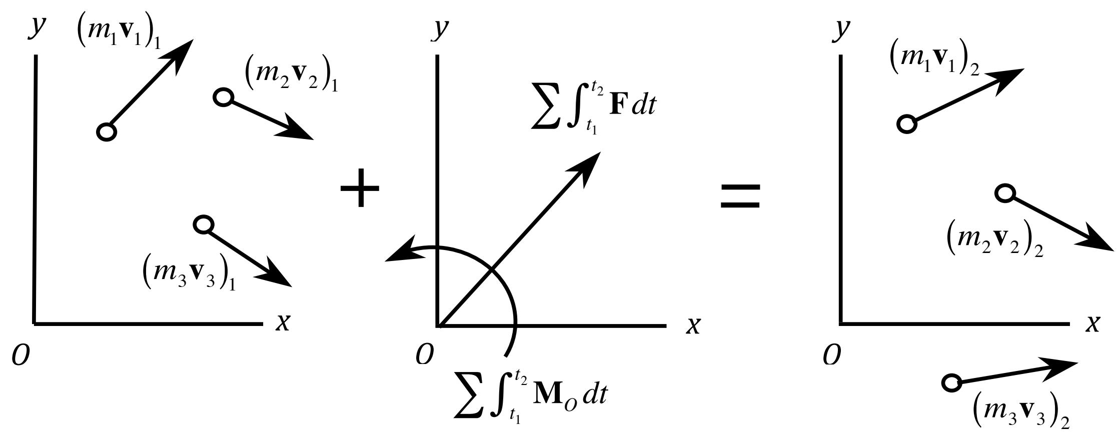
Note: Angular impulses from *internal* forces of action and reaction cancel.

If no external forces act from time t_1 to t_2, then

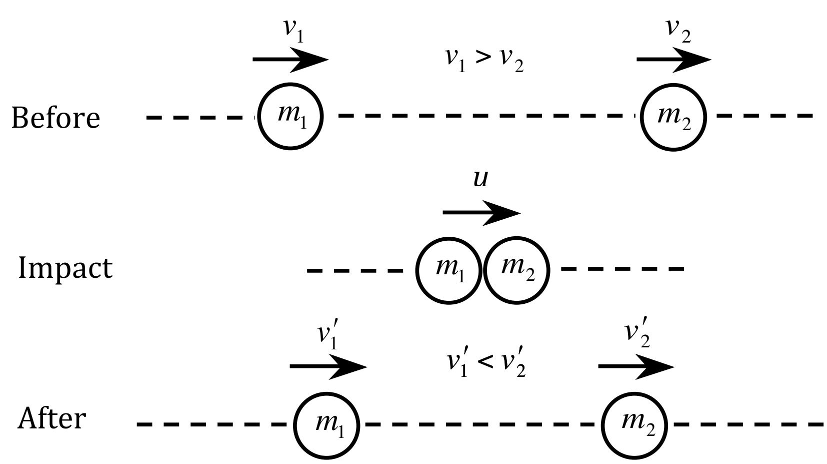
$$\sum(\mathbf{H}_O)_1 = \sum(\mathbf{H}_O)_2$$

and the *total angular momentum* of the particles is *conserved*.

Graphical interpretation:



Direct Central Impact



Total linear momentum is conserved during impact:

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$$

$$\textit{Coefficient of restitution}: \; e = \frac{\text{velocity of separation}}{\text{velocity of approach}} = \frac{v_2' - v_1'}{v_1 - v_2}$$

If total kinetic energy is conserved, impact is said to be *perfectly elastic* and $e = 1$.

If particles stick together after impact, $v_1' = v_2'$, impact is said to be *perfectly plastic*, and $e = 0$.

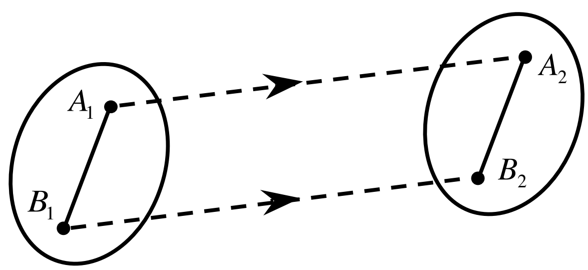
For all other impact cases, $0 \le e \le 1$.

A *special case* occurs when $m_1 = m_2$, collision is *elastic*, $v_1 > 0$, and $v_2 = 0$. Then, $v_1' = 0$ and $v_2' = v_1$.

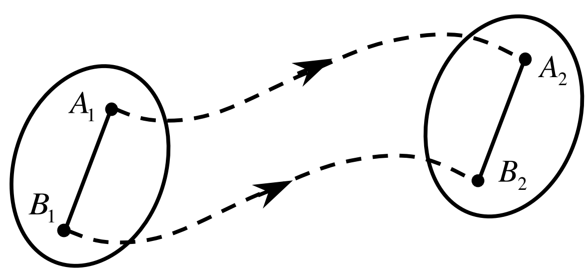
Kinematics of Rigid Bodies

Types of plane motion:

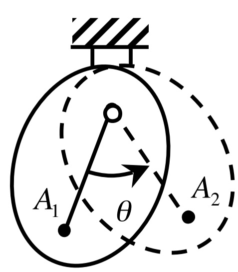
Rectilinear translation



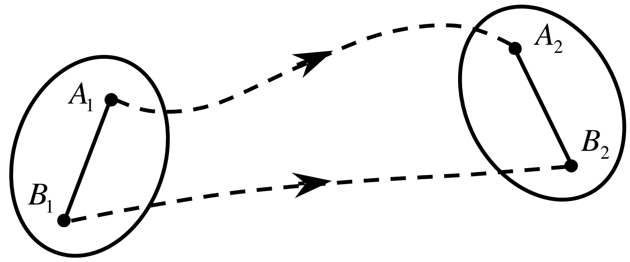
Curvilinear translation



Fixed-axis rotation



General plane motion



Combination of translation and rotation

Translation

Recall analysis of "Motion Relative to Translating Reference Axes":

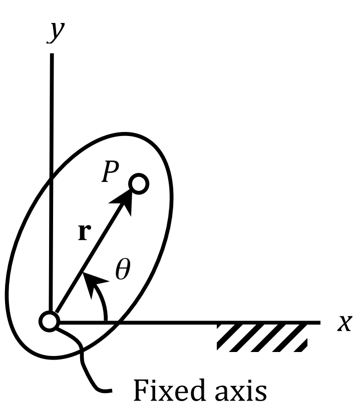
$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

Now A and B are any two particles in the translating rigid body. Therefore, $\mathbf{r}_{A/B} =$ **constant vector**, and

$$\mathbf{v}_A = \mathbf{v}_B$$
$$\mathbf{a}_A = \mathbf{a}_B$$

Rotation About a Fixed Axis

Recall analysis of "Angular Motion of a Line":



$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Define angular velocity vector $\boldsymbol{\omega}$ and angular acceleration vector $\boldsymbol{\alpha}$ as follows:

$$\boldsymbol{\omega} = \omega\mathbf{k}$$
$$\boldsymbol{\alpha} = \alpha\mathbf{k}$$

Then, the velocity of particle P is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ and the acceleration is

$$\begin{aligned}\mathbf{a} &= \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}\end{aligned}$$

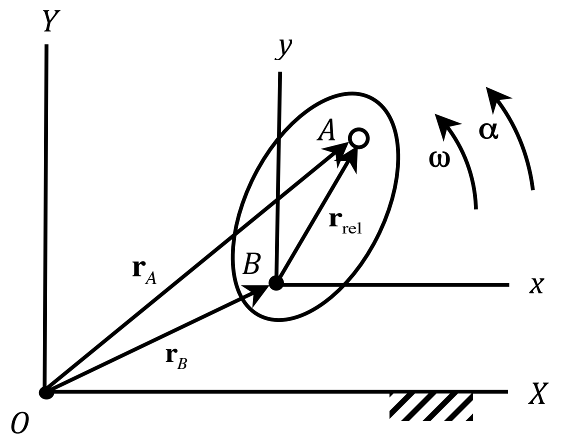
Note: In r - θ coordinates, $v_r = 0$ $\quad v_\theta = \omega r$

$a_r = -\omega^2 r \quad a_\theta = \alpha r$

In t - n axes, $v = \omega r$

$a_n = \omega^2 r \quad a_t = \alpha r$

General Plane Motion – Absolute and Relative Velocity and Acceleration



Axes $x-y$ translate with their origin attached to particle B.

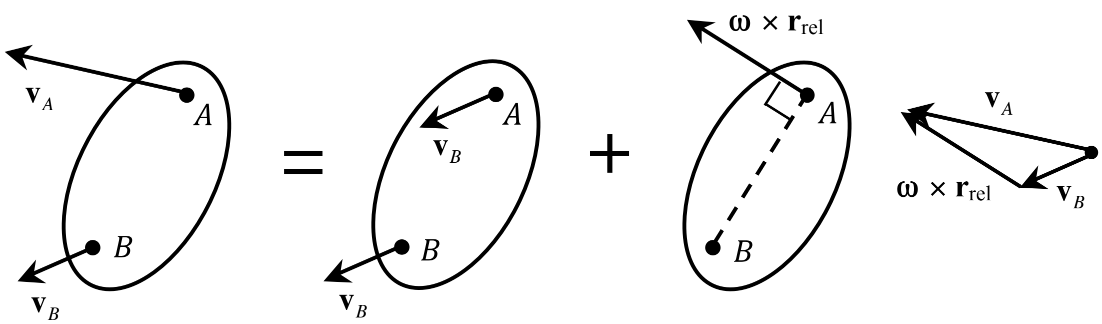
$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{\text{rel}}$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{\text{rel}}$$

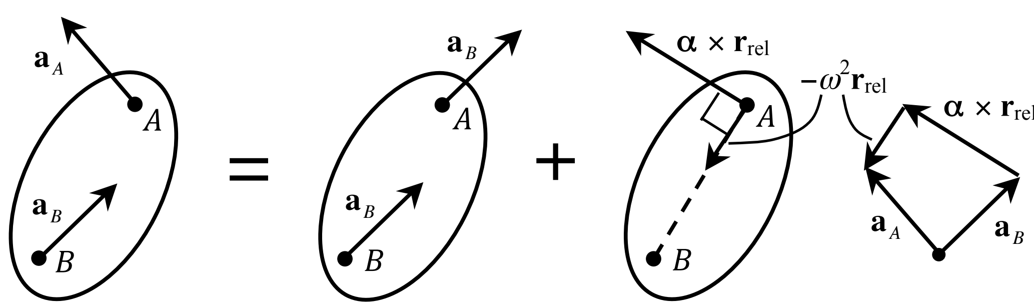
$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{\text{rel}})$$

$$= \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{\text{rel}} - \omega^2 \mathbf{r}_{\text{rel}}$$

Graphical interpretation:



Plane Motion $\quad$ Translation with $\mathbf{v}_B$ $\quad$ Rotation about B with $\boldsymbol{\omega}$



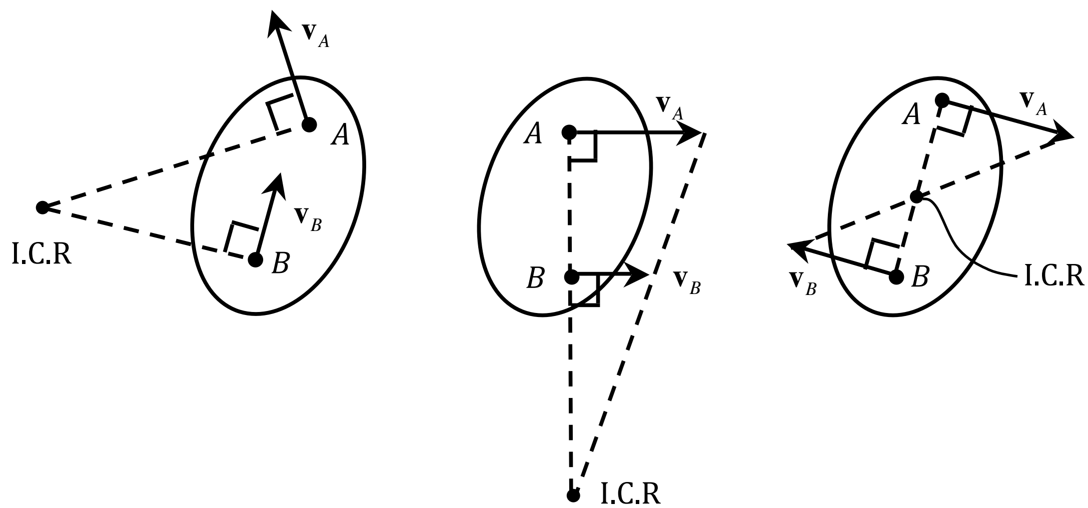
Plane Motion $\quad$ Translation with $\mathbf{a}_B$ $\quad$ Rotation about B with $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$

Instantaneous Center of Rotation in Plane Motion

Suppose $\mathbf{v}_B = \mathbf{0}$ in the previous analysis. Then,

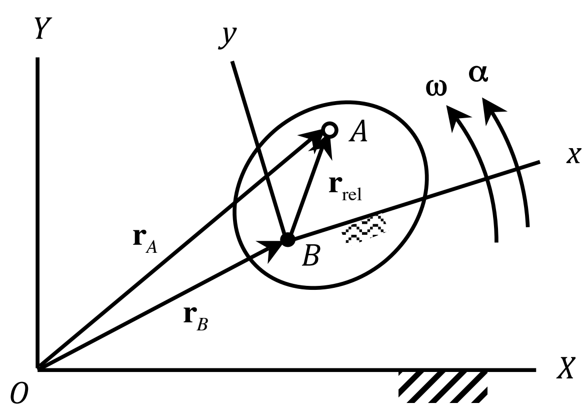
$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{\text{rel}}.$$

This result implies the body is rotating for an *instant* about point B. Such a point is called an *instantaneous center of rotation* (I.C.R.). Such a point can be determined, as follows, if the velocities of two different particles in a body are known.



Note: The location of the I.C.R. changes with time in general. Hence, $\mathbf{a}_{\text{ICR}} \neq \mathbf{0}$ in general!

Plane Motion of a Particle Relative to a Rotating Frame



Axes $x - y$ are body-fixed axes, which have angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$.

Particle A moves relative to the body-fixed axes $x - y$. The relative position vector of A referenced to the $x - y$ axes is

$$\mathbf{r}_{\text{rel}} = x\mathbf{i} + y\mathbf{j}$$

The *relative velocity* of A with respect to the $x-y$ axes is:

$$\mathbf{v}_{\text{rel}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$$

The *relative acceleration* of A with respect to the $x-y$ axes is:

$$\mathbf{a}_{\text{rel}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$$

The *absolute position vector* of A in the $X-Y$ inertial axes is given by:

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{\text{rel}}$$

The *absolute velocity* of A in the $X-Y$ inertial axes is given by:

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}}$$

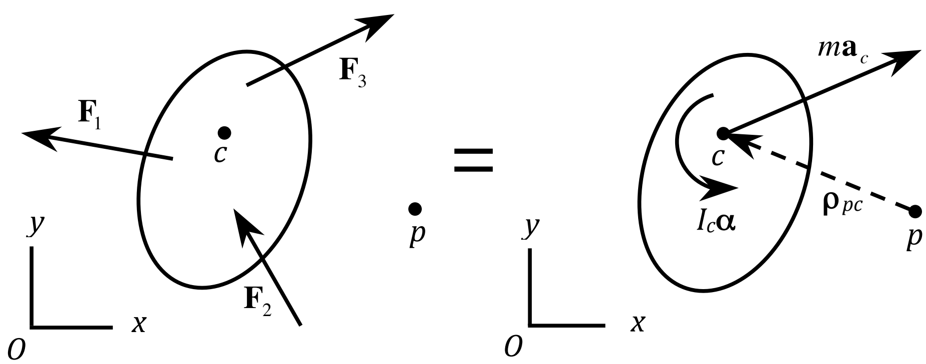
The *absolute acceleration* of A in the $X-Y$ inertial axes is given by:

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{\text{rel}}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

The term $2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}}$ is known as *Coriolis acceleration.*

Kinetics of Rigid Bodies: Forces and Accelerations

Equations of Motion for Body in Plane Motion



FBD KD

$$\sum \mathbf{F} = m\mathbf{a}_c$$

$$\sum \mathbf{M}_c = I_c\boldsymbol{\alpha}$$

or $$\sum \mathbf{M}_p = I_c\boldsymbol{\alpha} + \boldsymbol{\rho}_{pc} \times m\mathbf{a}_c$$

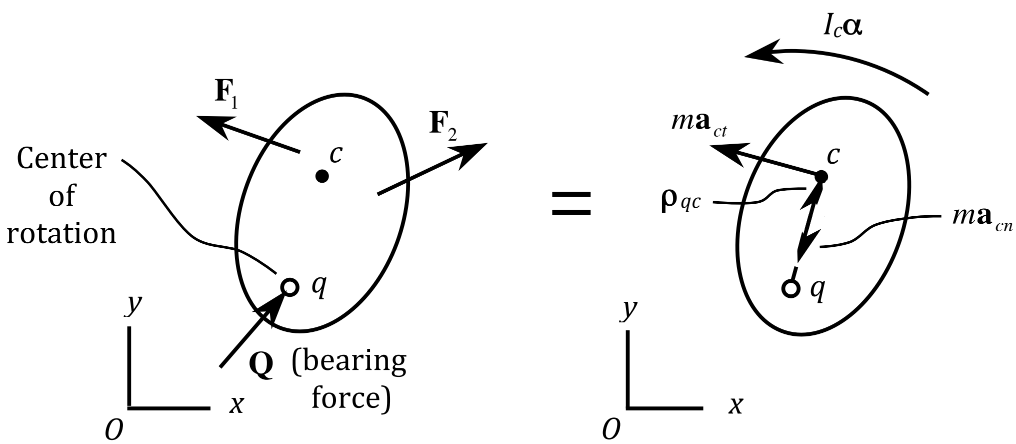
where: m = total mass
c = center of mass
I_c = mass moment of inertia about axis through c parallel to z-axis
p = any moment center in $x - y$ plane

In component form: $\sum F_x = ma_{cx}$

$$\sum F_y = ma_{cy}$$

$$\circlearrowleft \sum M_c = I_c\alpha$$

<u>Noncentroidal Rotation</u>



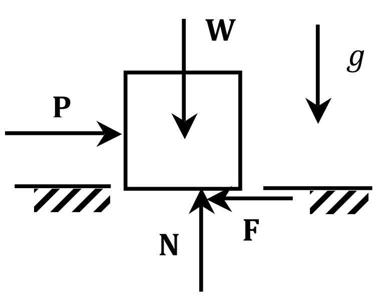
FBD KD

$$\sum \mathbf{M}_q = I_c\boldsymbol{\alpha} + \boldsymbol{\rho}_{qc} \times m\mathbf{a}_{ct}$$

$$= I_q\boldsymbol{\alpha}$$

where I_q = mass moment of inertia about axis through q parallel to *z-axis.*

<u>Laws of Friction</u>



Block is initially at rest when force **P** is applied and its magnitude is progressively increased from zero. As long as

$$P = F < \mu_s N,$$

the block will *not* slide.

μ_s = *coefficient of static friction*.

When $P = F = \mu_s N$, the block starts to slide, and F becomes:

$$F = \mu_k N$$

where $\mu_k =$ *coefficient of kinetic friction*,
$\mu_k < \mu_s$.

Kinetics of Rigid Bodies: Energy Methods

For a body in plane motion, the work done on the body by all external forces $\mathbf{F}_i$ is

$$U_{1\to 2} = \sum \int_{(\mathbf{r}_i)_1}^{(\mathbf{r}_i)_2} \mathbf{F}_i \cdot d\mathbf{r}_i$$

when the body is displaced from position 1 to position 2.

For a body in plane motion, the kinetic energy is

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$$

For a body in plane motion, the work done on the body by a couple M is

$$U_{1\to 2} = \int_{\theta_1}^{\theta_2} M\, d\theta$$

when the body is displaced from position 1 to position 2.

In general, $U_{1\to 2} = \frac{1}{2} m (v_c)_2^2 + \frac{1}{2} I_c \omega_2^2 - \left[\frac{1}{2} m (v_c)_1^2 + \frac{1}{2} I_c \omega_1^2 \right]$

$= T_2 - T_1$
$= \Delta T$

or $T_2 = T_1 + U_{1\to 2}$

If a gravitational force $\mathbf{W}$ acts on the body, and/or a linearly-elastic spring force, then the work-energy equation can be written as:

$$U_{1\to 2} = \Delta T + \Delta V_g + \Delta V_e$$

where $U_{1\to 2}$ now excludes gravitational and spring forces. If $U_{1\to 2}$ above is zero, *total mechanical energy is conserved.*

Noncentroidal Rotation

$T = \frac{1}{2} I_q \omega^2$ where q is the center of rotation.

Power developed by a couple **M**

$$\text{Power} = M\omega$$

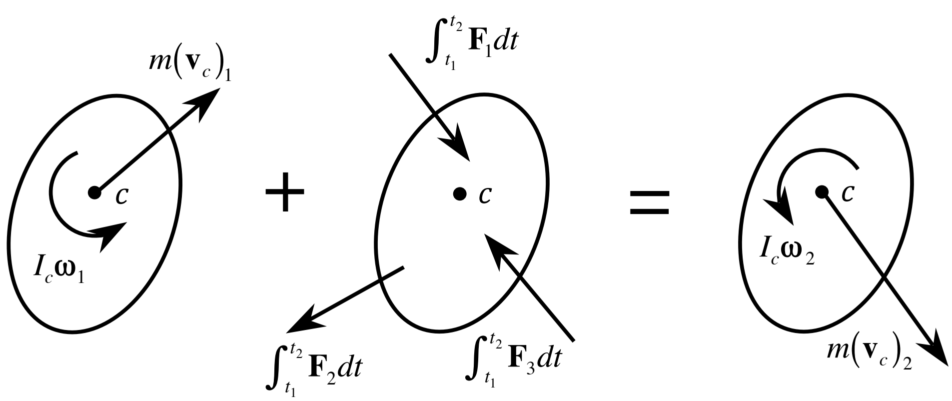
Kinetics of Rigid Bodies: Momentum Methods

For a body in plane motion, the equations of impulse and momentum are;

$$m(\mathbf{v}_c)_1 + \sum \int_{t_1}^{t_2} \mathbf{F}_i \, dt = m(\mathbf{v}_c)_2$$

$$I_c \boldsymbol{\omega}_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_c \, dt = I_c \boldsymbol{\omega}_2$$

Graphical interpretation



Initial linear and angular momenta

Sum of all linear and angular impulses (about c) from external forces

Final linear and angular momenta

If $\sum \int_{t_1}^{t_2} \mathbf{F}_i dt = \mathbf{0}$, then $m(\mathbf{v}_c)_1 = m(\mathbf{v}_c)_2$ and we say *linear momentum is conserved.*

If $\sum \int_{t_1}^{t_2} \mathbf{M}_p dt = \mathbf{0}$ about some point p, then

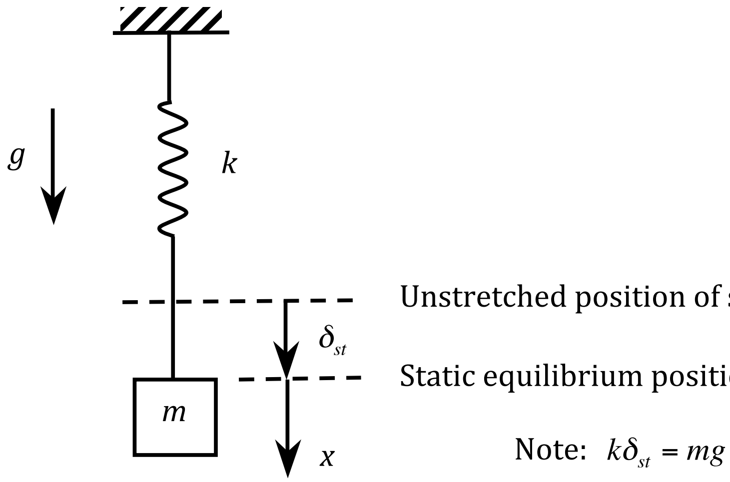
$$I_c \boldsymbol{\omega}_1 + \boldsymbol{\rho}_{pc} \times m(\mathbf{v}_c)_1 = I_c \boldsymbol{\omega}_2 + \boldsymbol{\rho}_{pc} \times m(\mathbf{v}_c)_2$$

and we say *total angular momentum about point p is conserved.*

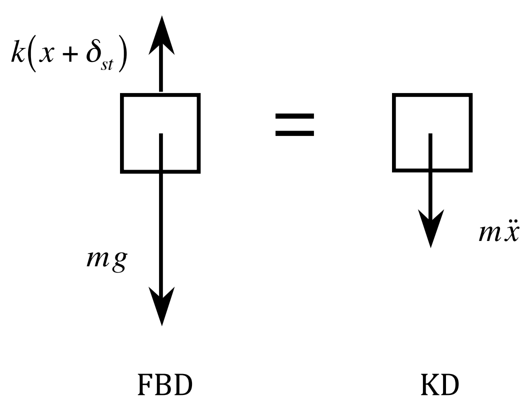
Vibrations

When a mass moves back and forth about an equilibrium position, the motion is described as *vibration*.

A simple example of vibration is the motion of a mass m connected to a massless spring with spring constant k.



Mass m displaced from its equilibrium position



$$\sum F_x = m\ddot{x}$$

$$-k(x + \cancel{\delta_{st}}) + \cancel{mg} = m\ddot{x}$$

$$m\ddot{x} + kx = 0 \quad (1) \quad \text{(Equation of motion)}$$

Let $\omega_n^2 = \dfrac{k}{m}$, then Eq. (1) becomes:

$$\ddot{x} + \omega_n^2 x = 0 \quad (2)$$

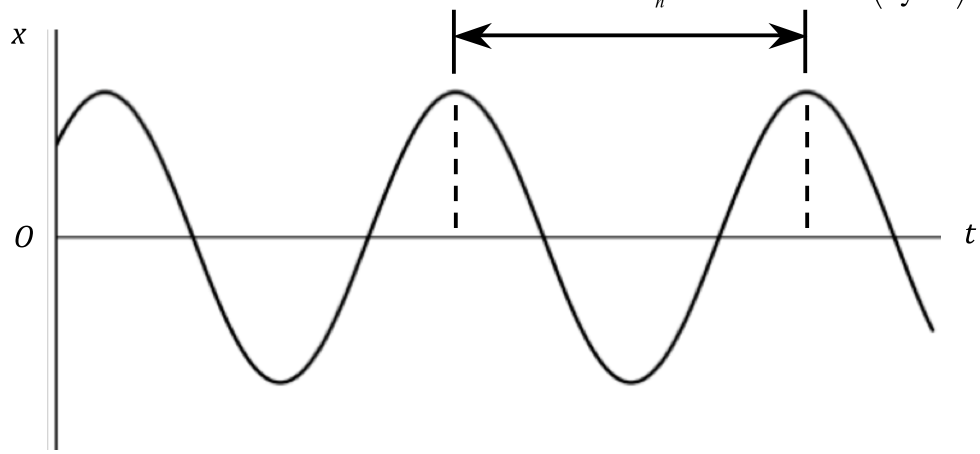
The solution of Eq. (2) is

$$x = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t)$$

where $x(0)$ and $\dot{x}(0)$ are initial conditions.

The motion is called *simple harmonic motion*, and $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$ is known as the *natural (circular) frequency* (rad/s).

$\tau_n = \frac{2\pi}{\omega_n}$, called *period* $\left(\frac{s}{\text{cycle}}\right)$



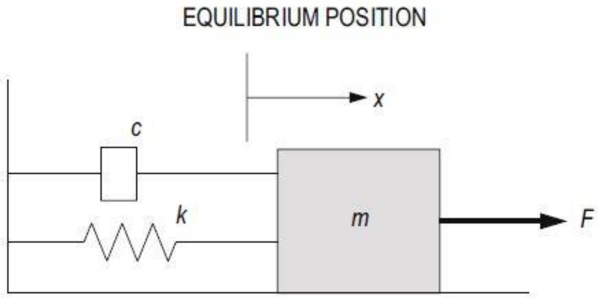
Since no forcing function appears in the equation of motion (1), the vibration above is called *free vibration*.

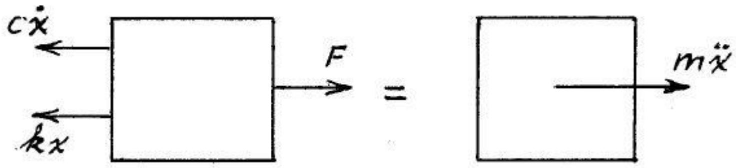
For an example of torsional vibration, see p. 127 in the 10.2 Handbook.

The following material is adapted from *FE Reference Handbook* 10.2

Free and Forced Vibration

A single degree-of-freedom vibration system, containing a mass m, a spring k, a viscous damper c, and an external applied force F can be diagrammed as shown:





FBD KD

The equation of motion for the displacement of x is:

$$m\ddot{x} = -kx - c\dot{x} + F$$

or in terms of x,

$$m\ddot{x} + c\dot{x} + kx = F$$

One can define

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

$$K = \frac{1}{k}$$

Then:

$$\frac{1}{\omega_n^2}\ddot{x} + \frac{2\zeta}{\omega_n}\dot{x} + x = KF$$

If the externally applied force is 0, this is a free vibration, and the motion of x is solved as the solution to a homogeneous ordinary differential equation.

In a forced vibration system, the externally applied force F is typically periodic (for example, $F = F_0 \sin \omega t$). The solution is the sum of the homogeneous solution and a particular solution.

For forced vibrations, one is typically interested in the steady state behavior (i.e. a long time after the system has started), which is the particular solution.

For $F = F_0 \sin \omega t$, the particular solution is:

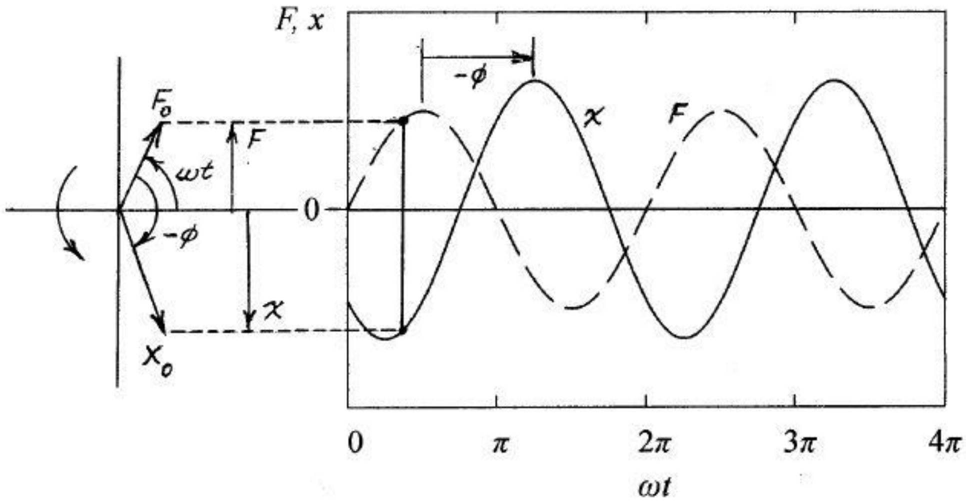
$$x(t) = X_0 \sin(\omega t + \phi)$$

where

$$X_0 = \frac{KF_0}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

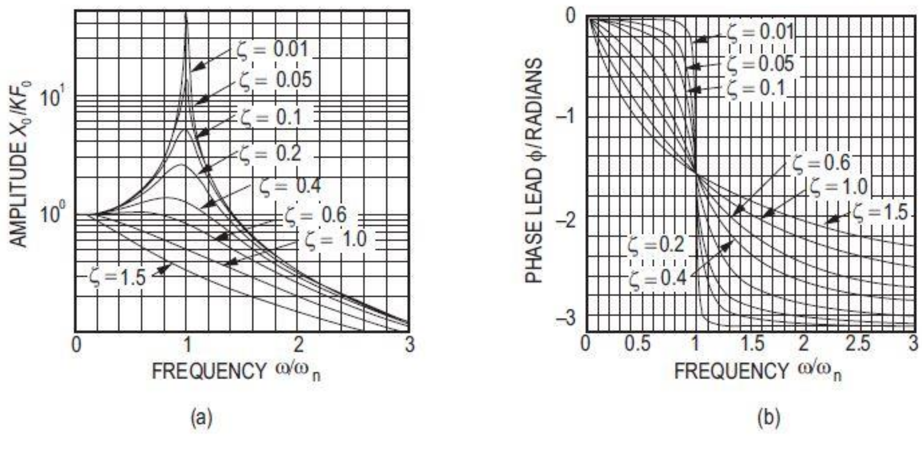
$$\phi = \tan^{-1}\frac{-\frac{2\zeta\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

An example of a particular solution is:



$\omega > \omega_n$, $\phi = -3\pi/4$

The following figures provide illustrative plots of relative amplitude and phase, depending on ω and ω_n.



Steady state vibration of a force spring-mass system (a) amplitude (b) phase.

From Brown University School of Engineering, Introduction to Dynamics and Vibrations, as posted on www.brown.edu/Departments/Engineering/Courses/En4/Notes/vibrations_forced/vibrations_forced.htm, April 2019.