ON MINIMUM SPANS OF NO-HOLE T-COLORINGS

Daphne Der-Fen Liu*
and
Roger K. Yeh**

April 24, 1999

Abstract. We study a variation of T-coloring which arose from the channel assignment problem introduced by Hale [5]. Given a non-negative integral set T containing 0, a T-coloring on a graph G is a function that assigns to each vertex a non-negative integer (color) under the restriction that the difference of colors of any two adjacent vertices does not fall in T. A no-hole T-coloring is a T-coloring such that the colors used must be consecutive. Roberts [14] and Sakai and Wang [15] studied no-hole T-colorings for $T = \{0, 1\}$ and $\{0, 1, 2, \ldots, r\}$, respectively. We explore this problem for other sets T. A characterization of the existence of no-hole T-colorings is obtained immediately by combining graph homomorphism and a special family of graphs named T-graphs. For given T and G, the consecutive T-span, $\text{csp}_T(G)$, is the minimum span among all possible no-hole T-colorings of G if there exists one; otherwise $\text{csp}_T(G) = \infty$. If G has a no-hole T-coloring, then $|V(G)| - 1$ is an upper bound of $\text{csp}_T(G)$. We show that the upper bound is attained by large T-graphs if $T = \{0, 1, \cdots, r\} \cup A$ where A contains no multiples of $(r + 1)$, or $T = \{0, a, a + 1, \cdots, b\}$.

1. Introduction

T-Colorings arose from the channel assignment problem introduced by Hale [5], in which one non-negative integer (channel) is assigned to each radio station or transmitter so that interference is avoided. Interference occurs when the separation of channels of two close locations falls within the given integral set T (called T-set) containing 0. A graphical model can be constructed for this problem as follows. Let each station be represented by a vertex, and make an edge between two vertices that correspond to close stations. Thus for a given graph $G(V, E)$ and T-set, a *T-coloring* (or a valid channel assignment) of G is a function f: $V(G) \to Z^+ \cup \{0\}$ such that if $\{u, v\} \in E(G)$ then $|f(u) - f(v)| \notin T$. The T-coloring problem has been studied extensively (See [2 - 4, 8 - 13, 16].)

---

*Corresponding author. Department of Mathematics and Computer Science, California State University, Los Angeles, USA. E-mail: dliu@calstatela.edu. Research supported in part by a CSU Research and Scholarship grant and by the National Science Foundation under grant DMS 9805945.

**Department of Applied Mathematics, Feng Chia University, Taiwan. E-mail: rkyeh@math.fcu.edu.tw. Research supported in part by the National Science Council of Taiwan under grant NSC-82-0208-M035-010.

Typeset by $\mathcal{AMS}$-TeX

In this article, we study a variant of the channel assignment problem in which the channels used must be consecutive. We call a T-coloring f *no-hole T-coloring* if $f(V)$ is a set of consecutive integers.

In terms of the efficient use of the channels, span is one of the variables that have been studied most extensively in the channel assignment problem. The *span* of a T-coloring f is the difference of the largest and smallest colors used, *i.e.*, $\max_{u,v\in V}|f(u)-f(v)|$. For given T and G, the *T-span* of G, denoted by $\mathrm{sp}_T(G)$, is the minimum span over all possible T-colorings of G. We define the *consecutive T-span*, $\mathrm{csp}_T(G)$, by the minimum T-span among all no-hole T-colorings of G if there exists one; otherwise, $\mathrm{csp}_T(G)=\infty$. Notice that for a given T-set, not every graph has a no-hole T-coloring. For instance, if $1\in T$, any complete graph with more than one vertex does not have a no-hole T-coloring.

Given T and G, if G has a no-hole T-coloring, then an immediate upper bound for $\mathrm{csp}_T(G)$ is $|V(G)|-1$. And since any no-hole T-coloring is also a T-coloring, we have:

$$\mathrm{sp}_T(K_{\omega(G)}) \leqslant \mathrm{sp}_T(G) \leqslant \mathrm{csp}_T(G) \leqslant |V(G)|-1, \qquad (*)$$

where $\omega(G)$ is the smallest number of vertices of a complete graph (clique) in G and K_n is a clique on n vertices. The first inequality in (*) is straightforward.

Introducing the notion of no-hole T-colorings, Roberts [14] studied the no-hole T-coloring with $T=\{0,1\}$. Sakai and Wang [15] studied the no-hole T-coloring with $T=\{0,1,2,\dots,r\}$. The exact values of $\mathrm{csp}_T(G)$ as $T=\{0,1,2,\dots,r\}$ for bipartite graphs were discussed in [1]. It was shown in [14], among other results, that when $T=\{0,1\}$, G is a path, cycle, or 1-unit sphere graph, and G has a no-hole T-coloring, then $\mathrm{csp}_T(G)\leqslant \mathrm{sp}_T(G)+1$. Similar results for $T=\{0,1,2,\dots,r\}$ were presented in [15], namely, if $T=\{0,1,2,\dots,r\}$ and G satisfies the same assumptions as above, then $\mathrm{csp}_T(G)\leqslant \mathrm{sp}_T(G)+r$. A natural question raised in [14] states: If $T=\{0,1\}$, is it true that $\mathrm{csp}_T(G)\leqslant \mathrm{sp}_T(G)+1$ for any graph that has a no-hole T-coloring ?" The answer is negative. It was proved in [15] that there exist graphs G such that $\mathrm{sp}_T(G)+r<\mathrm{csp}_T(G)<\infty$, when $T=\{0,1,2,\dots,r\}$.

We study the consecutive T-spans of T-graphs for two more general families of T-sets. Given T, the *T-graph* denoted by G_T is defined by $V(G_T)=\mathcal{Z}^+\cup\{0\}$, and $E(G_T)=\{ab : |a-b|\notin T\}$. The T-graph of order n, denoted by G_T^n, is the subgraph of G_T induced by the first n vertices of G_T, $\{0,1,2,\dots,n-1\}$. For two integers a and b, we denote $[a,b]$ the set of consecutive integers $\{a,a+1,a+2,\dots,b\}$. Given any positive integer r, a T-set is *r-initial* if $T=[0,r]\cup S$, where S contains no multiple of $(r+1)$. We show that if T is r-initial or $T=\{0\}\cup[a,b]$, then there exists N so that for any $n\geqslant N$, $\mathrm{csp}_T(G_T^n)=n-1$.

2. Preliminaries

There is a direct characterization of the existence of a no-hole T-coloring by using graph homomorphism and T-graphs. For two simple graphs G and H, a *graph homomorphism* from G to H is a function $f: V(G)\to V(H)$ such that if u and v are adjacent in G, then $f(u)$ and $f(v)$ are adjacent in H. If such a function exists, we say that G is homomorphic to H and denote this by $G\to H$. Furthermore, if the homomorphism is onto, then it is an *epimorphism* and denoted by $G \xrightarrow{\text{onto}} H$.

For a graph G, the *chromatic number*, $\chi(G)$, is the minimum number of colors in a proper vertex-coloring (*i.e.* adjacent vertices receive different colors). The following are basic properties about the relationships between graph homomorphism, $\chi(G)$, $\omega(G)$, and $\mathrm{sp}_T(G)$. The proofs can be obtained by composition of functions.

Proposition 2.1. *If $G \to H$, the following are all true:*
(a) $\chi(G) \leqslant \chi(H)$,
(b) $\omega(G) \leqslant \omega(H)$,
(c) for any T-set, $\mathrm{sp}_T(G) \leqslant \mathrm{sp}_T(H)$.

Notice that if G is homomorphic to H, the inequality $\mathrm{csp}_T(G) \leqslant \mathrm{csp}_T(H)$ is not always true. For instance, if $T = \{0, 1, 4, 5\}$, $G = K_2$, and $H = C_5$ (five-cycle), then $G \to H$, and $\{0, 1, 2, 3, 4\}$ can be assigned to the vertices of H as a no-hole T-coloring, so $\mathrm{csp}_T(H) \leqslant 4$, however, $\mathrm{csp}_T(G) = \infty$. If we add a condition that the homomorphism from G to H is onto, that is $G \xrightarrow{\text{onto}} H$, then any no-hole T-coloring of H can be transformed into a no-hole T-coloring of G by a composition of functions, so $\mathrm{csp}_T(G) \leqslant \mathrm{csp}_T(H)$. That is,

Proposition 2.2. *If $G \xrightarrow{\text{onto}} H$, then for any T, $\mathrm{csp}_T(G) \leqslant \mathrm{csp}_T(H)$.*

The following characterization of the existence of a no-hole T-coloring can be obtained directly from definition.

Theorem 2.3. *Given T, a graph G has a no-hole T-coloring with span $k - 1$ if and only if $G \xrightarrow{\text{onto}} G_T^k$.*

Corollary 2.4. *Given T and G, if $\chi(G) > \chi(G_T^k)$ or $\omega(G) > \omega(G_T^k)$, then $\mathrm{csp}_T(G) > k-1$.*

Proof. Suppose G has a no-hole T-coloring with span $k' - 1$, $k' \leqslant k$. By Theorem 2.3, $G \xrightarrow{\text{onto}} G_T^{k'}$. Referring to Prop. 2.1, $\chi(G) \leqslant \chi(G_T^{k'}) \leqslant \chi(G_T^k)$ and $\omega(G) \leqslant \omega(G_T^{k'}) \leqslant \omega(G_T^k)$. □

Given T and G, the calculation of $\mathrm{csp}_T(G)$ or determining whether there is a no-hole T-coloring of G in general is difficult. In view of Theorem 2.3, the existence of a no-hole T-coloring is equivalent to the existence of an epimorphism from G to a T-graph with finite order. It has been proved by Hell and Nešetřil [7] that determining whether there is a homomorphism from G to H in general is NP-complete unless G is bipartite; when G is bipartite, then it is polynomial. Via private communication, Hell [6] claimed that determining whether there is an epimorphism from G to H in general is NP-complete for all non-bipartite graphs G as well as for some bipartite graphs G.

3. Consecutive T-spans of T-graphs

In this section, we prove that $\mathrm{csp}_T(G_T^n)$ equals the upper bound $n-1$, provided n large for r-initial sets and $T = \{0\} \cup [a, b]$.

For any T-set and positive integer n, the enumeration of the vertices of G_T^n, $[0, n-1]$, itself is a no-hole T-coloring. Hence $\mathrm{csp}_T(G_T^n) \leqslant n - 1$. If n is an integer such that

$\mathrm{sp}_T(K_m) = n-1$ for some m, then $K_m \to G_T^n$, so $\omega(G_T^n) \geqslant m$. By (*), this implies $n-1 \leqslant \mathrm{sp}_T(K_{\omega(G_T^n)}) \leqslant \mathrm{sp}_T(G_T^n) \leqslant \mathrm{csp}_T(G_T^n) \leqslant n-1$, so $\mathrm{csp}_T(G_T^n) = n-1$. It is known [8] that $\mathrm{sp}_T(K_m) = n-1$ if and only if n is the smallest such that $\omega(G_T^n) = m$. Thus we have

Lemma 3.1. *Given T, if $\omega(G_T^n) > \omega(G_T^{n-1})$, then* $\mathrm{csp}_T(G_T^n) = n-1$.

If the given T-set is T', we call a T-coloring of G a T'-coloring, and denote the consecutive T-span of G by $\mathrm{csp}_{T'}(G)$, and the T-graph of T' by $G_{T'}$.

Lemma 3.2. *If $T \subseteq T'$, then for any G and $n \in Z^+$, the following are all true:*
(a) $\mathrm{csp}_T(G) \leqslant \mathrm{csp}_{T'}(G)$,
(b) $\mathrm{csp}_T(G_{T'}^n) \leqslant \mathrm{csp}_T(G_T^n) \leqslant \mathrm{csp}_{T'}(G_T^n)$,
(c) $\mathrm{csp}_T(G_{T'}^n) \leqslant \mathrm{csp}_{T'}(G_{T'}^n) \leqslant \mathrm{csp}_{T'}(G_T^n)$.

Proof. (a) Since $T \subseteq T'$, any no-hole T'-coloring of G is also a no-hole T-coloring of G, so the inequality holds.

The second inequality in (b) and the first inequality in (c) can be derived from (a). The remaining inequalities of (b) and (c) can be obtained from Prop. 2.2 and the fact that $G_{T'}^n \xrightarrow{\text{onto}} G_T^n$. □

Note that $\mathrm{csp}_{T'}(G_T^n)$ in the lemma above may not be finite. For instance, let $T = \{0, 1\}$ and $T' = \{0, 1, 2\}$, then $\omega(G_T^5) = 3 > \omega(G_{T'}^5) = 2$. By Corollary 2.4, $\mathrm{csp}_{T'}(G_T^5) > 4$ which is impossible, so $\mathrm{csp}_{T'}(G_T^5) = \infty$.

Now we prove the sharpness of the upper bound of $\mathrm{csp}_T(G_T^n)$ for the *largest* r-initial set, $T_{r+1} = \{0\} \cup \{x \in \mathcal{Z}^+ : x \text{ is not a multiple of } (r+1)\}$.

Theorem 3.3. *Let $T = T_{r+1}$, then*

$$\mathrm{csp}_T(G_T^n) = \begin{cases} 0, & 1 \leqslant n \leqslant r+1; \\ n-1, & n \geqslant r+2. \end{cases}$$

Proof. If $n \leqslant r+1$, then G_T^n is an independent graph (*i.e.* no edge) so $\mathrm{csp}_T(G_T^n) = 0$.

If $n \geqslant r+2$, it is straightforward to verify that G_T^n is a disjoint union of $r+1$ cliques, $G_T^n = S_0 \uplus S_1 \uplus S_2 \uplus \cdots \uplus S_r$, where $V(S_i) = \{q(r+1)+i : q \in Z^+ \cup \{0\} \text{ and } q(r+1)+i < n\}$.

Suppose $\mathrm{csp}_T(G_T^n) = k-1 \leqslant n-2$. Let $f : V(G_T^n) \to [0, k-1]$ be a no-hole T-coloring of G_T^n. Because T contains all positive integers except multiples of $(r+1)$, the colors used in any clique S_i must have the same residue modulo $(r+1)$. Without loss of generality, we may assume that for any $0 \leqslant i \leqslant r$ and $u, v \in V(S_i)$, $f(u) \equiv f(v) \equiv i \pmod{(r+1)}$. Since f is onto and $k < n$, there exist vertices x and y in G_T^n such that $f(x) = f(y)$, so $f(x) \equiv f(y) \pmod{(r+1)}$, but x and y must belong to different cliques. This is a contradiction, thus $\mathrm{csp}_T(G_T^n) = n-1$. □

To prove the sharpness of the upper bound of $\mathrm{csp}_T(G_T^n)$ for r-initial sets, we also need the following lemma.

Lemma 3.4. *Let $T = [0, r]$, then*

$$\mathrm{csp}_T(G^n_{T_{r+1}}) = \begin{cases} 0, & 1 \leqslant n \leqslant r+1; \\ n-1, & n \geqslant r+2. \end{cases}$$

Proof. From the proof of Theorem 3.3, $G^n_{T_{r+1}} = S_0 \uplus S_1 \uplus S_2 \uplus \cdots \uplus S_r$. It suffices to show the case for $n \geqslant r+2$, *i.e.* $|S_0| \geqslant 2$. Since $G^n_{T_{t+1}} \xrightarrow{\text{onto}} G^n_T$, $\mathrm{csp}_T(G^n_T) \leqslant n-1$. Suppose $\mathrm{csp}_T(G^n_{T_{r+1}}) = k-1$ for some $k < n$. Let $f : V(G^n_{T_{r+1}}) \to [0, k-1]$ be a no-hole T-coloring of $G^n_{T_{r+1}}$. Then there exist vertices u and v such that $f(u) = f(v)$. Let $u \in S_i$, $v \in S_j$ and $i \neq j$. It is enough to examine the following two cases:

Case 1. $|S_i| = |S_j| = 1$. Since $|S_0| \geqslant 2$, there exist x and y in S_0 with $|f(x) - f(y)| \geqslant r+1$ ($\because T = [0, r]$). Without loss of generality, assume $f(x) < f(y)$. Because f is onto, the r colors $[f(x)+1, f(x)+r]$ must be used in the remaining r cliques, $S_1, S_2, \ldots, S_r$, and no two of them can be assigned to the same clique. This is impossible since $f(u) = f(v)$.

Case 2. $|S_i|$ (or $|S_j|$) $\geqslant 2$. Then there exists $x \in S_i$, $x \neq u$, and $|f(u) - f(x)| \geqslant r+1$. If $f(u) = u_0 < f(x)$, then these r colors $[u_0+1, u_0+r]$ have to be used in the remaining r cliques, and no two of them can be assigned to the same clique. Because $f(v) = f(u) = u_0$, none of those r colors can be assigned to S_j, a contradiction. Similarly, if $f(u) = u_0 > f(x)$, then one can use the r colors $[u_0 - 1, u_0 - r]$ to obtain a contradiction. □

Theorem 3.5. *If T' is r-initial, then*

$$\mathrm{csp}_{T'}(G^n_{T'}) = \begin{cases} 0, & 1 \leqslant n \leqslant r+1; \\ n-1, & n \geqslant r+2. \end{cases}$$

Proof. We only have to show the case for $n \geqslant r+2$. Let $T = [0, r]$ and $T'' = T_{r+1}$, then $T \subseteq T' \subseteq T''$. By Lemma 3.4, $\mathrm{csp}_T(G^n_{T''}) = n-1$ as $n \geqslant r+2$. Since $\mathrm{csp}_{T'}(G^n_{T'}) \leqslant n-1$, it is enough to show that $\mathrm{csp}_T(G^n_{T''}) \leqslant \mathrm{csp}_{T'}(G^n_{T'})$. By Lemma 3.2, $\mathrm{csp}_T(G^n_{T'}) \leqslant \mathrm{csp}_{T'}(G^n_{T'})$. Because $G^n_{T''} \xrightarrow{\text{onto}} G^n_{T'}$, by Prop. 2.2, $\mathrm{csp}_T(G^n_{T''}) \leqslant \mathrm{csp}_T(G^n_{T'})$. Therefore, $\mathrm{csp}_T(G^n_{T''}) \leqslant \mathrm{csp}_{T'}(G^n_{T'})$ which completes the proof. □

In a T-graph G_T, the *recursive clique of size i*, denoted by RK_i, is the clique with vertex set defined by $V(RK_1) = \{0\}$, and for $m \geqslant 2$, $V(RK_m) = V(RK_{m-1}) \cup \{x\}$ where x is the first vertex adjacent to all $V(RK_{m-1})$. As $n \to \infty$, denote the *infinite recursive clique* in G_T by RK. It has been proved [10] that if $T = \{0\} \cup [a, b]$, then for any $n \in Z^+$ the maximum recursive clique in G^n_T is also a maximum clique. To prove the sharpness of the upper bound of $\mathrm{csp}_T(G^n_T)$ for the family $T = \{0\} \cup [a, b]$, we make use of the following lemma.

Lemma 3.6. *Let $T = \{0\} \cup [a, b]$, $b = ka + i$, $0 \leqslant i \leqslant a-1$. If f is a no-hole T-coloring of G with span $m-1$, where $m \leqslant (k+1)a + i - 1$, then f must assign consecutive colors to any K_a in G.*

Proof. Suppose f is a no-hole T-coloring of G with span $m-1$. Let K_a be an a-clique in G. Let $0 \leqslant s = \min f(K_a)$, and suppose $\{s, s+1, s+2, \ldots, x\} \subseteq f(K_a)$ for some

$s \leqslant x \leqslant s+a-2$, but $x+1 \notin f(K_a)$. Because $[a,b] \subseteq T$, the rest of the colors that may be assigned to K_a are in the set $B = [b+x+1, m-1]$. Since $|B| = m-1-(ka+i+x) \leqslant a-2-x$, we have $|f(K_a)| \leqslant x+1+a-2-x = a-1$, which contradicts the fact that K_a requires a different colors.

Theorem 3.7. *Let $T = \{0\} \cup [a,b]$, $b = ka+i$ and $0 \leqslant i \leqslant a-1$, then*

$$\mathrm{csp}_T(G_T^n) = \begin{cases} a-1, & a+1 \leqslant n \leqslant (k+1)a; \\ n-1, & \textit{otherwise.} \end{cases}$$

Proof. Since the first a vertices in G_T form a clique, $\mathrm{csp}_T(G_T^n) = n-1$ for $n \leqslant a$. For $a+1 \leqslant n \leqslant (k+1)a$, it suffices to show $\mathrm{csp}_T(G_T^{(k+1)a}) = a-1$, since $a-1 \leqslant \mathrm{csp}_T(G_T^{a+1}) \leqslant \mathrm{csp}_T(G_T^{(k+1)a})$. Define a modular coloring $g : V(G_T^{(k+1)a}) \to [0, a-1]$ by $g(x) = (x \bmod a)$. Then g is onto. We show that g is a T-coloring. If x and y are adjacent in G_T^n, then either $|x-y| \leqslant a-1$ or $|x-y| \geqslant b+1 = ka+i+1 \equiv i+1 \pmod a$. In the former case, $g(x) \neq g(y)$; in the latter case, $g(x) = g(y)$ only when $i = a-1$. This is impossible, since then $|x-y| \geqslant k(a+1)$, but $V(G_T^{(k+1)a}) = [0, k(a+1)-1]$. Therefore g is a no-hole T-coloring and $\mathrm{csp}_T(G_T^n) = a-1$ for $a+1 \leqslant n \leqslant (k+1)a$.

In G_T, the vertex set of the infinite recursive clique RK consists of the following periodic intervals of vertices, *i.e.*, all the intervals have the same length, and the separations of any two consecutive intervals are the same:

$$V(RK) = [0, a-1] \cup [b+a, b+2a-1] \cup [2b+2a, 2b+3a-1] \cup \ldots\ldots$$

In the representation above, we call the set of vertices within one interval a *period* which consists of a consecutive numbers. The first set of integers is the first period and so on. The known result that the maximum recursive clique in G_T^n for any n is also a maximum clique [10] implies that $\omega(G_T^n)+1 = \omega(G_T^{n+1})$ for all $n \in V(RK)$. By Lemma 3.1, $\mathrm{csp}_T(G_T^{n+1}) = n$ for all $n \in V(RK)$.

Between the second and third periods, it is enough to show that $\mathrm{csp}_T(G_T^n) = n-1$ for $b+2a+1 \leqslant n \leqslant 2b+2a$. We claim the following two cases, the remaining ones can be obtained similarly.

Claim 1: $\mathrm{csp}_T(G_T^{2a+b+1}) = 2a+b$

Assume to the contrary, $\mathrm{csp}_T(G_T^{2a+b+1}) < 2a+b$. Then $\mathrm{csp}_T(G_T^{2a+b+1}) = 2a+b-1$, since $\omega(G_T^{2a+b+1}) = \omega(G_T^{2a+b}) > \omega(G_T^{2a+b-1})$ and $\mathrm{csp}_T(G_T^{2a+b}) = 2a+b-1$. By Theorem 2.3, there exists an epimorphism $f : G_T^{2a+b+1} \to G_T^{2a+b}$. Because f is onto, there is only one color to be used exactly twice and other colors once. We call this *one repetition.*

In G_T, since $n = 2a+b$ is the smallest number such that $\omega(G_T^n) = 2a$, f can only use the $2a$ colors $[0, a-1] \cup [a+b, 2a+b-1]$ on any K_{2a}. There are two K_{2a} in G_T^{2a+b+1}: A_1 and A_2 where $A_1 = [0, a-1] \cup [a+b, 2a+b-1]$ and $A_2 = [1, a] \cup [a+b+1, 2a+b]$. Thus, $|A_1 \cup A_2| = 2a+2$ but $|f(A_1 \cup A_2)| \leqslant 2a$, so at least two repetitions are necessary for f on $A_1 \cup A_2$, a contradiction. ■

Claim 2: $\mathrm{csp}_T(G_T^{2a+b+2}) = 2a+b+1$

Assume $\mathrm{csp}_T(G_T^{2a+b+2}) < 2a+b+1$, then there are the following two possibilities:

(i) Suppose $\mathrm{csp}_T(G_T^{2a+b+2}) = 2a+b$, then by Theorem 2.3 there is an epimorphism $f : G_T^{2a+b+2} \to G_T^{2a+b+1}$ with exactly one repetition. There are only two K_{2a} in G_T^{2a+b+1}: $[0, a-1] \cup [a+b, 2a+b-1]$ and $[1, a] \cup [a+b+1, 2a+b]$, so on any K_{2a}, f can only use the colors from $[0, a] \cup [a+b, 2a+b]$. There are three K_{2a} in G_T^{2a+b+2}:

$$A_1 = [0, a-1] \cup [a+b, 2a+b-1]$$
$$A_2 = [1, a] \cup [a+b+1, 2a+b]$$
$$A_3 = [2, a+1] \cup [a+b+2, 2a+b+1].$$

Let $A = A_1 \cup A_2 \cup A_3$, then $|A| = 2a+4$, while $|f(A)| \leqslant 2a+2$. This implies that at least two repetitions are necessary, a contradiction.

(ii) Suppose $\mathrm{csp}_T(G_T^{2a+b+2}) = 2a+b-1$, then there is an epimorphism $f : G_T^{2a+b+2} \to G_T^{2a+b}$ with exactly two repetitions. On any K_{2a}, $f(K_{2a}) = [0, a-1] \cup [a+b, 2a+b-1]$ since $[0, a-1] \cup [a+b, 2a+b-1]$ is the only K_{2a} in G_T^{2a+b}. Hence, for the A defined in (i) above, $|f(A)| = 2a$ but $|A| = 2a+4$, so at least four repetitions are necessary, a contradiction. ■

Analogously, between the m-th and the (m+1)-th periods for $m \geqslant 3$, one can apply an argument similar to the above by replacing the clique K_{2a} by K_{ma}.

Between the first and second periods, one needs to show that $\mathrm{csp}_T(G_T^n) = n-1$ for $(k+1)a+1 \leqslant n \leqslant a+b = (k+1)a+i$. Letting $n = (k+1)a+s$ with $1 \leqslant s \leqslant i$, we prove this by induction on s.

(Initial step) $s = 1$. We need to show $\mathrm{csp}_T(G_T^{(k+1)a+1}) = (k+1)a$. Since any consecutive a vertices in G_T form a clique, the modular coloring g defined at the beginning of the proof of the theorem is the only proper coloring for $G_T^{(k+1)a}$ in a colors, so $\chi(G_T^{(k+1)a}) = a$. Furthermore, because 0 is adjacent to $(k+1)a$, we have $\chi(G_T^{(k+1)a+1}) = a+1$. Therefore, it is impossible to have $G_T^{(k+1)a+1} \xrightarrow{\text{onto}} G_T^n$ for $n \leqslant (k+1)a$, thus $\mathrm{csp}_T(G_T^{(k+1)a+1}) = (k+1)a$.

(Inductive step) Assuming $\mathrm{csp}_T(G_T^{n-1}) = n-2$, we need to show $\mathrm{csp}_T(G_T^n) = n-1$ for $n = (k+1)a+s$, $2 \leqslant s \leqslant i$. If $a = 1$ or 2, we are done. For if $a = 1$, then T is r-initial; if $a = 2$, then i is either 0 or 1. Thus, we may assume $a \geqslant 3$. Suppose $\mathrm{csp}_T(G_T^n) = n-2$ and let f be a no-hole T-coloring $f : V(G_T^n) \to [0, n-2]$ with exactly one repetition. Denote $f(j)$ by x_j for $0 \leqslant j \leqslant n-1$. By Lemma 3.6, any K_a in G_T^n must use consecutive colors, and any consecutive a vertices in G_T^n form a clique, thus the color 0 can only be used exactly once and $x_j \neq 0$ for $j \in [2, n-3]$. Otherwise, there will be more than one repetition.

If $x_0 = 0$ (or $x_{n-1} = 0$), then $f|_{G_T^n - \{0\}}$ (or $f|_{G_T^n - \{n-1\}}$, respectively) is a no-hole T-coloring with span $n-3$, which implies $\mathrm{csp}_T(G_T^{n-1}) \leqslant n-3$ ($\because G_T^n - \{0\}$ is isomorphic to G_T^{n-1}), contradicting the inductive hypothesis $\mathrm{csp}_T(G_T^{n-1}) = n-2$.

If $x_1 = 0$, then the two cliques $[0, a-1]$ and $[1, a]$ can only use the colors $[0, a-1]$. Thus, $x_0 = x_a \in [1, a-1]$ is the only repetition in f. Since there is no other repetition allowed and any K_a must use consecutive a colors, we have $x_{a+i} = a+i-1$, $1 \leqslant i \leqslant n-a-1$, and $x_0 = x_a = a-1$. Because $n = (k+1)a+s \geqslant (k+1)a+2$, 0 is adjacent to $n-1$. But $x_{n-1} - x_0 = ka+s-1 \in T$, a contradiction. Similarly, one can show $x_{n-2} \neq 0$. Therefore, $0 \notin f(V)$, a contradiction which claims $\mathrm{csp}_T(G_T^n) \neq n-2$.

Suppose $\text{csp}_T(G_T^n) = n-i$, $i \geqslant 3$, and let f be a no-hole T-coloring $f : V(G_T^n) \to [0, n-i]$ with exactly $i-1$ repetitions. Hence $x_j \neq 0$ for $j \in [i, n-i-1]$. If $x_0 = 0$ (or $x_{n-1} = 0$), then $\text{csp}_T(G_T^{n-1}) \leqslant n-3$, contradicting the inductive hypothesis. If $x_0 = x_j$ (or $x_{n-1} = x_j$) for some j, then $\text{csp}_T(G_T^{n-1}) \leqslant n-3$, a contradiction. Thus colors x_0 and x_{n-1} are used exactly once. This implies $x_j \neq 0$ for $j = 1, 2 \ldots, i-1, n-i, n-i+1, \ldots, n-2$, so $0 \notin f(V)$, a contradiction. Therefore, $\text{csp}_T(G_T^n) \neq n-i$, $i \geqslant 3$. □

Thus far, we have shown for two families of T-sets, $\text{csp}_T(G_T^n) = n-1$ for large n, however, a characterization of T-sets with this property is still unknown. The following is an example of T-set that does not have this property.

Example. *If* $T = (Z^+ \cup \{0\}) - [1, r]$, $r \in Z^+$, *then*

$$csp_T(G_T^n) = \begin{cases} n-1, & n \leqslant r+1; \\ r, & n \geqslant r+2. \end{cases}$$

Proof. In G_T, the first $r+1$ vertices form a clique, so $\text{csp}_T(G_T^n) = n-1$ for $n \leqslant r+1$, and $\text{csp}_T(G_T^n) \geqslant r$ for $n \geqslant r+1$. Define a modular coloring $f : V(G_T^n) \to \{0, 1, 2, \ldots, r\}$ by $f(u) \equiv (u \bmod (r+1))$, then f is a no-hole T-coloring. Hence $\text{csp}_T(G_T^n) = r$ for $n \geqslant r+1$. □

References

1. G. Chang, S. Juan and D. Liu, *No hole $(r+1)$-distance coloring for bipartite graphs*, manuscript.
2. M. B. Cozzens and F. S. Roberts, *T-colorings of graphs and the channel assignment problem*, Congressus Numerantium **35** (1982), 191-208.
3. M. B. Cozzens and F. S. Roberts, *Greedy algorithms for T-colorings of complete graphs and the meaningfulness of conclusions about them*, J. Comb. Inform. Syst. Sci. **16** (1991), 286-299.
4. J. R. Griggs and D. D.-F. Liu, *The channel assignment problem for mutually adjacent sites*, J. Comb. Theory A **68** (1994), 169-183.
5. W. K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE **68** (1980), 1497-1514.
6. P. Hell, private communication.
7. P. Hell and J. Nešetřil, *On the Complexity of H-coloring*, Journal of Combinatorial Theory, Series **B** **48** (1990), 92 - 110.
8. D. D.-F. Liu, *T-coloring of graphs*, Discrete Math **101** (1992), 203 - 212.
9. D. D.-F. Liu, *On a conjecture of T-colorings*, Congr. Numer **103** (1994), 27-31.
10. D. D.-F. Liu, *T-Graphs and the channel assignment problem*, Discrete Math **161** (1996), 197-205.
11. J. H. Rabinowitz and V. K. Proulx, *An asymptotic approach to the channel assignment problem*, SIAM J. Alg. Disc. Math. (1985), 507-518.
12. A. Raychaudhuri, *Further results on T-coloring and frequency assignment problem*, SIAM J. Disc. Math. **7** (1994), 605-613.
13. F. S. Roberts, *T-colorings of graphs: recent results and open problems*, Disc. Math. **93** (1991), 229-245..
14. F. S. Roberts, *No-hole 2-distant colorings*, Mathl. Comput. Modelling **17** (1993), 139-144.
15. D. Sakai and C. Wang, *No-hole $(r+1)$-distant colorings*, Discrete Math **119** (1993), 175-189.
16. B. A. Tesman, *T-colorings, List T-colorings and set T-Colorings of Graphs*, Ph.D. Dissertation, Dept. Math., Rutgers University (1988).