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Mill's *A System of Logic*

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3 Mill and the Philosophy of Mathematics

Physicalism and Fictionalism

Mark Balaguer

I will do three things in this chapter. First, in section 1, I will provide a brief description of Mill's philosophy of mathematics. Then in section 2, I will explain why Mill's view can't account for contemporary mathematics or even the mathematics of his own day. Finally, in section 3, I will explain what Mill *should have* said about mathematics, given his background philosophical commitments.

Before starting, I should say that the section 1 presentation of Mill's view will be fairly brief. I'm not going to try to bring out all of the different features of his view. My aim is to present just the central core of his view. This will be enough to set up the arguments in section 2 for the claim that Mill's view isn't right.

I. MILL'S PHILOSOPHY OF MATHEMATICS

In order to understand Mill's philosophy of mathematics, you first have to understand what motivates him to say what he says. What motivates him are his background commitments to a thoroughgoing empiricist epistemology and a naturalistic worldview, i.e., a physicalistic, anti-platonist worldview. These background views *seem* to be incompatible with the discipline of mathematics. For (a) mathematics doesn't seem to be an empirical science—it seems to be an *a priori* science—and (b) it doesn't seem to be giving us theories of the physical world; rather, it seems to be giving us theories of platonistic mathematical structures like the natural-number sequence. So mathematics is a problem case for Mill. He needs to give an account of mathematics to show that it isn't a counterexample to his overall philosophical view.

In what follows, I will mostly be concerned with Mill's view of arithmetic, but I want to start by saying a few words about his view of geometry. To this end, let's consider a simple geometrical sentence that Mill discusses:

(R) All of the radii of a circle are equal in length.

What kinds of objects is this sentence *about*? Well, obviously, it's about circles and radii. But what kinds of objects are they? One natural response to this question is that circles and radii are *platonic* objects, or abstract objects—i.e., non-physical, non-mental, non-spatiotemporal objects. But given Mill's background philosophical commitments, it seems that he can't say this. So what he says instead is that ordinary geometrical sentences like (R) are about *physical* things—e.g., real circles drawn on a page with pen and ink. Now, since no physical circle is a *perfect* circle—or put differently, since no physical object satisfies the mathematical *definition* of a circle—Mill is forced to say that ordinary geometrical sentences like (R) are *not strictly speaking true*. In fact, the point here isn't simply that (R) isn't true of *all* circles; according to Mill, the sentence (R) ". . . is not exactly true of any circle [i.e., any real, physical circle]; it is only *nearly* true[.]"[1]

As long as we read (R) as being about physical circles, as opposed to platonistic circles, this point is entirely obvious. If I draw a circle on a page, and if I then proceed to draw several radii of that circle, then no matter how careful I am, it simply won't be true that all the radii I draw will be *exactly* equal in length; they will only be *nearly* equal in length. So on Mill's view, sentences like (R) are strictly speaking *false*.

It is important to note, however, that Mill does think that a certain sort of necessity attaches to geometrical sentences like (R). In particular, he thinks that they follow of necessity from the definitions and axioms of geometry. Here is Mill assenting to this view:

> When, therefore, it is affirmed that the conclusions of geometry are necessary truths, the necessity consists in reality only in this, that they correctly follow from the suppositions from which they are deduced.[2]

Mill is here denying that the theorems of geometry are necessary truths about real objects; there may be some necessary truths in the vicinity of geometry, but they aren't the theorems themselves, as ordinarily stated; they're *conditionals*—e.g., sentences like the following:

> (R#) If the axioms of geometry are true, and if there are objects answering to the definitions of geometry (e.g., if there are objects that really satisfy the precise definition of a perfect circle), then speaking of these perfect circles, we can say that (R) is strictly and literally true—i.e., we can say that all of the radii of a (perfect) circle are (exactly) equal in length.

On Mill's view, sentences like (R#) are not just true, but necessarily true. But this is perfectly consistent with a physicalistic, empiricist worldview; for (R) is *analytic*; its antecedent analytically entails its consequent.

Let me move on now to Mill's view of arithmetic. And, again, let me begin by looking at what Mill says about a simple arithmetical sentence. Here's one that he discusses:

(A) 2 + 1 = 3.

Prima facie, this sentence seems to be a claim about three numbers—namely, 1, 2 and 3—but, once again, this face-value reading seems to be off limits to Mill, given his background philosophical commitments. For, *prima facie*, it seems that if there are any such things as numbers, then they're abstract objects; and it seems that Mill can't allow that sentences like (A) are about abstract objects. Thus, to avoid claiming that (A) is literally about *numbers*, Mill takes it to be a general claim about all objects—or more precisely, all *collections* of objects, or *piles* of objects, or some such thing.[3] Mill puts the point as follows:

> All numbers must be numbers of something; there are no such things as numbers in the abstract. *Ten* must mean ten bodies, or ten sounds, or ten beatings of the pulse. But though numbers must be numbers of something, they may be numbers of anything. Propositions, therefore, concerning numbers have the remarkable peculiarity that they are propositions concerning all things whatever, all objects, all existences of every kind known to our experience.[4]

If we take the sentence (A) as our example, then we can read Mill as saying (roughly—more on this below) that (A) is equivalent to the following sentence:

> (A-M) For any two (disjoint, i.e., non-overlapping) piles of objects x and y, if there are two objects in x and if there's one object in y, then the pile that's made up of x and y together has three objects in it.

For instance, to use an example that Mill likes to use, the sentence (A) tells us, among other things, that if we push two pebbles together with one pebble, we will have three pebbles.

This gives Mill the desired result that arithmetical sentences like (A) are about ordinary physical objects, or piles of objects, rather than numbers. And it also gives him a way of claiming that sentences like (A) are *empirical*; for on Mill's view, these sentences are universal generalizations about piles of physical objects, and we come to know that these sentences are true by *induction*—in particular, from the many experiences that we have with ordinary piles of physical objects from early childhood on. Mill assents to this view in a number of places. For instance, after saying that the real fact behind (A) is that any three-membered pile of objects can be rearranged into a two-membered pile and a singleton pile, he says:

> The Science of Number is thus no exception to the conclusion previously arrived at that the processes even of deductive sciences are altogether inductive and that their first principles are generalizations from experience.[5]

So Mill's view of arithmetic is similar to his view of geometry in that he takes both branches of mathematics to be concerned with *physical* objects, and in both cases, he thinks that our knowledge is ultimately based in *empirical* knowledge of the relevant physical objects.

But does Mill think that the claims of arithmetic are strictly speaking false, in the way that those of geometry are? It depends. If we take a sentence like (A) and we apply it to things like *weight*—i.e., if we use it to calculate the weight of a pile of one-pound weights—then (A) will be only approximately true. For no physical object that goes by the name "one-pound weight" will actually weigh exactly one pound. So if we push one of these things together with two of them and infer that the whole pile weighs three pounds, this will be only approximately true. But as long as we're talking about just the *number* of things in a pile, as opposed to something like their weight, then arithmetical claims can be exactly true, according to Mill. For instance, if I have (exactly) two pebbles in my left hand and (exactly) one pebble in my right hand, then (assuming I have only two hands) I have *exactly* three pebbles in my hands.

Given this stance of Mill's, I think we have to conclude that it's not quite right to say that according to Mill, (A) is equivalent to (A-M). On his reading, (A) is even more general than (A-M); it entails (in a way that (A-M) doesn't) that two pounds and one pound make three pounds. This consequence of (A) isn't strictly speaking true, but it's still a consequence of that sentence, on Mill's view. In any event, in what follows, I will ignore this complication and write as if Mill takes (A) to be equivalent to (A-M). This will simplify the discussion, and no harm will come from simplifying in this way. In particular, none of the objections that I raise against Mill's view relies in any important way on this simplification.

There is a lot more that could be said about Mill's philosophy of mathematics; the remarks I've made here give only a brief introduction to his view. But these remarks are enough to see how Mill thinks that someone with his background philosophical views (i.e., someone who is committed to a naturalistic, anti-platonistic ontology and an empiricist epistemology) can account for mathematics. And as we'll presently see, the brief remarks I've provided here are also already enough to bring out the fact that Mill's view of mathematics is untenable.

II. PROBLEMS WITH MILL'S VIEW

Let me start by talking about Mill's view of arithmetic. The above remarks already make clear that Mill is committed to a certain view of the semantics of the language of arithmetic. In particular, he is committed to the idea that the sentences of arithmetic are about physical objects. I think this theory is clearly false. To see why, consider the following sentence, which was proved by Euclid:

(B) There are infinitely many prime numbers.

I'm not aware of Mill discussing sentences like this anywhere, but he seems to be committed to taking such sentences to be about physical objects, or piles of physical objects, or some such thing. After all, we presumably want to endorse a uniform semantics for all of arithmetic. So if sentences like (A) are to be read as being about physical objects, then we're going to have to read sentences like (B) in this way as well. But the problem is that there doesn't seem to be any plausible way to take (B) to be about physical objects. To see why, imagine a mathematics professor teaching Euclid's proof of sentence (B) to a classroom full of students, and imagine a student (say, Beatrice) raising her hand with the following objection:

> There couldn't be infinitely many prime numbers because my physics professor told me that there are only finitely many physical objects in the whole universe.

It seems reasonable to think that Beatrice just *doesn't understand*; she doesn't understand what Euclid's proof is supposed to show. For in the context of Euclid's proof, it doesn't *matter* how many physical objects there are. Even if it's true that there are only finitely many physical objects in the universe, this is simply *not* a good reason to reject Euclid's proof. And the only reasonable conclusion we can draw from this, I think, is that Euclid's theorem—i.e., sentence (B)—should not be interpreted as being about physical objects. It has to be read as being about something else; for if it *were* about physical objects, then Beatrice's worry would *not* be misguided. Thus, since Beatrice's worry *is* misguided—since it involves a failure to understand what Euclid's theorem is *about*—we have to conclude that Euclid's theorem is not about physical objects.

This problem with Mill's view is made more vivid if we switch to set theory. Now, you might think it's unfair to ask whether Mill's view can handle the claims of set theory, since set theory wasn't developed until long after Mill developed his view; but I disagree with this. First of all, the last edition of *A System of Logic* didn't appear until 1872. But the main point I want to make here is that I don't think the chronology matters. Set theory is an important branch of mathematics, so I think it's important to see whether the general Millian approach to mathematics can handle the development of this new theory. So let's consider the following theorem of set theory, which was proven by Cantor in 1870:

(C) There are infinitely many transfinite cardinals that keep getting bigger and bigger without end.

Once again, it's hard to see how we can interpret this sentence as being about physical objects. Imagine a professor proving this theorem in class, and imagine a student (say, Big Red) complaining that this so-called "theorem" couldn't really be true because there aren't enough physical objects in the whole universe to make it true. It seems clear that what we ought to say about Big Red is that he *doesn't understand*; in particular, he doesn't understand what (C) *says*. For in the context of Cantor's theorem, the question of how many physical objects there are in the universe is entirely irrelevant. And given this, we have no choice but to infer that Cantor's theorem is simply not about physical objects.

A second problem with Mill's view is that it's out of touch with the actual methodology of actual working mathematicians. Mill's theory implies that the right methodology for determining whether sentences like (B) and (C) are true would involve an empirical investigation into the number of physical objects in the universe. But this means that it implies that the proofs of Euclid and Cantor are not just mistaken but completely wrongheaded in their methodology. If Mill were right, then Euclid and Cantor should have used empirical methods. But, of course, this is crazy. There's nothing wrong with the methodology of mathematical proof; the problem here is that Mill's view is simply false.

Very quickly, here's a third argument against Mill's view: when we apply his physicalistic approach to set theory, we get the conclusion that expressions that are supposed to refer to sets should be interpreted as referring to piles of physical objects. But this can't be right, because corresponding to every pile of physical objects—indeed, to every individual physical object—there are infinitely many sets. Corresponding to a ball, for instance, is the set containing the ball, the set containing its molecules, the set containing its atoms, and so on. Moreover, in addition to the set containing, say, the atoms, there's also the set containing that set, the set containing that set and so on to infinity. Clearly, these sets are not supposed to be purely physical objects, because (a) they are all supposed to be numerically distinct from one another, and (b) they all share the same physical base (i.e., the same pile of matter and the same spatiotemporal location). Thus, if these sets exist at all, then there must be something non-physical about them, over and above the physical base that they all share—i.e., the physical matter that makes up the ball. So sets cannot be purely physical objects. Or to put the point in semantic terms, the singular terms of set theory are not supposed to refer to purely physical objects. And the problem is that it's hard to see how we can accommodate this result without completely abandoning the Millian approach to mathematics.

I think there's something very telling about the inability of the Millian philosophy of mathematics to provide a plausible account of set theory. For insofar as Mill takes arithmetic to be about *piles* of objects, it seems to me that if his view can't provide a tenable account of the mathematical theory of *sets* of objects, then there's something seriously wrong with his view.

Finally, let's turn to geometry. Mill's view of the semantics of the language of geometry is counterintuitive in the extreme. It just seems hard to believe that when we do geometry, we're talking about real (i.e., imperfect) squares and circles and so on, and that when we assert the theorems of geometry, we're asserting strictly false near-truths about physical objects. This just flies directly in the face of our semantic intuitions about what we're talking about when we do geometry. The much more natural line to take here, on the semantics of the language of geometry, is that we're talking about perfect squares and circles, regardless of whether they exist.

But rather than just relying on an appeal to our semantic intuitions, let me offer a quick argument against Mill's view of geometry. His view seems to be completely incompatible with our response to the development of non-Euclidean geometries. If Mill were right, then geometry would be an empirical science about actual physical space, and the question of whether we ought to adopt Euclidean geometry or some non-Euclidean geometry would be a difficult empirical question for physicists. But whereas there may well be a difficult empirical question here about the structure of physical space, that's a question for *applied* mathematics. It's got nothing at all to do with *pure* mathematics. As far as pure mathematics is concerned, all there is to say is that Euclidean geometry is a good theory of Euclidean geometrical spaces, and Lobachevskian geometry is a good theory of Lobachevskian geometrical spaces, and so on. This is how the mathematical community has in fact responded to the discovery of non-Euclidean geometries. And this is the *right* response. And the reason it's the right response is that insofar as a geometrical theory is a pure mathematical theory, it simply isn't about anything physical. It isn't about real physical circles, as Mill contended, and it isn't about actual physical space.

III. WHAT MILL SHOULD HAVE SAID ABOUT MATHEMATICS

I think there's another view that Mill could have endorsed in connection with mathematics that would have been consistent with both (a) his background philosophical commitments (i.e., naturalism, anti-platonism, empiricism, etc.) and (b) the actual facts about mathematical practice. To get at the view I've got in mind, let's start by making sure that we satisfy constraint (b)—i.e., by making sure we come up with a view that's consistent with actual mathematical practice. And let's begin here by trying to come up with the right semantic theory for ordinary mathematical discourse.

III.i. What's the Right Semantic Theory for the Language of Mathematics?

Mill took ordinary mathematical sentences to be about physical objects, and I have argued here that this view is untenable. My argument focused on sentences like (B) and (C)—i.e., on sentences about infinities—but given that we want a uniform semantics for the language of arithmetic, the argument applies equally to simple sentences like (A), i.e., to sentences like "2 + 1 = 3." So let's start by trying to figure out what we should say about sentences like this. Or to simplify things even more, let's switch to a simple sentence of the form "*Fa*"—i.e., to a sentence like

(D) 3 is prime.

Prima facie, this sentence seems to be of the form "*Fa*." In other words, it seems to have the same logical form as sentences like "Mars is round." Just as the latter sentence says that a certain object (namely, Mars) has a certain property (namely, roundness), so too (D) seems to say that a certain object (namely, the number 3) has a certain property (namely, primeness).

Mill presumably thinks that if we accepted this view, then as empiricists and naturalistic anti-platonists, we would be led immediately into trouble. And so he rejects the face-value reading of arithmetical sentences. Again, he interprets (A) not as a claim about the numbers 1, 2 and 3, but as a universal generalization about all objects, or piles of objects. This, as we've seen, was a mistake. But I think there's a prior mistake already inherent in the decision to reject the face-value reading of sentences like (A)–(D). To appreciate this, let's forget about mathematics for a minute and just think about how we ought to proceed when we're doing empirical semantics, i.e., when we're interpreting the speech of ordinary humans. In this connection, it seems to me that the following principle is extremely plausible:

> *Face-value-ism*: When we interpret people's speech, the default setting is always to take them to be speaking literally, or at face value. To motivate a non-face-value interpretation of a given sentence, we have to motivate the claim that the speaker or speakers in question have positive intentions to be saying something other than what the sentence says literally, i.e., something other than what the sentence says when read at face value.

When I say that the speakers in question have to have *positive* intentions to be saying something non-literal, I don't mean that they have to have *conscious* intentions to be saying something non-literal; all I mean is that there has to be something about their overall psychology that makes it the case that they mean to be saying something other than what the sentence says at face value. When this point is appreciated, Face-value-ism becomes something close to a truism. Suppose that when sentence S is read at face value, it says that P, and suppose further that when people utter S, they don't have *any* intention, conscious or unconscious, to be saying anything other than P; then it's hard to see how we could plausibly maintain that when people utter S, they're actually saying something other than P. Let me illustrate this with an example. To this end, consider the following two sentences:

(M1) Mars is round.
(M2) The average mother has 2.4 children.

Read at face value, both of these sentences seem to say that a certain object has a certain property. (M1) seems to say that the object Mars has the property of roundness, and (M2) seems to say that a different object (namely, the average mother) has the property of having 2.4 children. But, of course, this isn't really what (M2) says. Or at any rate, this isn't how typical utterances of sentences like (M2) should be interpreted; if someone actually uttered (M2), it would almost certainly be the case that what the person was actually saying was that *on average, mothers have 2.4 children*. So in connection with sentences like (M2), it's plausible to endorse a non-face-value interpretation. But this is *only* because we have good empirical reasons to think that when people utter sentences like (M2), they intend to be speaking non-literally, and they *don't* intend to be interpreted at face value, i.e., as speaking of weird objects like *the average mother*.

But the situation with respect to (M1) is entirely different. When people say things like this, they usually don't have any intention to be saying anything non-literal. Now, of course, we could construct a case in which someone used an utterance of (M1) to say just about anything—e.g., that the microfilm is hidden in the foie gras. But (a) this would be true only if the speaker in question had an intention to be saying something non-literal, and (b) in connection with *ordinary* utterances of sentences like (M1), where the speaker doesn't have any such intention, there is no plausible way to treat these utterances as saying something non-literal.

If Face-value-ism is right, then we need to ask the following question: when ordinary mathematicians and ordinary folk utter ordinary mathematical sentences like (A)–(D), do they have any intention to be speaking non-literally? The answer to this question is, I think, obvious; in ordinary cases, when people utter ordinary mathematical sentences, they do *not* have any intention to be speaking non-literally. Only someone who was worried about the philosophical implications of our mathematical utterances—someone like Mill—would ever have such an intention. And so it seems to me that we have strong reasons to think that ordinary utterances of mathematical sentences like (A)–(D) should be read at face value. Thus, if we focus on (D)—i.e., on the sentence "3 is prime"—it should be read as making a claim about the number 3; in particular, it should be read as saying that that object has the property of primeness.

Now, you might think that accepting this conclusion is incompatible with Mill's background philosophical commitments—i.e., with the adoption of a naturalistic, anti-platonistic metaphysics and an empiricist epistemology. But let's keep an open mind about this, and let's keep pursing the current line of thought.

If we endorse a face-value interpretation of the sentence "3 is prime," then the next question we need to ask is what the singular term "3" is supposed to refer to (or equivalently, what the sentence "3 is prime" is supposed to be about). There are three possible answers to this question:

Semantic physicalism: Numerals like "3" are supposed to refer to physical objects, and so ordinary arithmetical sentences like "3 is prime" are best interpreted as straightforward claims about physical objects.

Semantic psychologism: Numerals like "3" are supposed to refer to mental objects, presumably ideas in our heads, and so ordinary arithmetical sentences like "3 is prime" are best interpreted as straightforward claims about mental objects, i.e., things that exist in actual human heads.

Semantic platonism: Numerals like "3" are supposed to refer to abstract objects—i.e., non-physical, non-mental, non-spatiotemporal objects—and so ordinary arithmetical sentences like "3 is prime" are best interpreted as straightforward claims about abstract objects.

There are several problems with semantic physicalism. First, given that we're reading "3" as a singular term, i.e., a term that's supposed to denote a unique object, if we take it to refer to a physical object, then it's hard to see *which* physical object it could be referring to. It's not as if we're going to unearth the number 3 on an excavation in South Dakota. The problem here is that there is no specific physical object that could plausibly be taken as being the referent of the numeral 3. If we're going to endorse a physicalistic philosophy of arithmetic, the only reasonable way to proceed is the Millian way—i.e., to abandon the face-value interpretation of "3 is prime" and to read it as a general claim about all three-membered piles of physical objects. In addition to this, semantic physicalism has all the problems that Mill's view has. For instance, it seems that if semantic physicalism were true—if numerals were supposed to refer to ordinary physical objects—then it would be reasonable to worry that there aren't enough objects in the entire universe to serve as referents for all of the various numerals (or to make sentences like "There are infinitely many prime numbers" true). Thus, since this is in fact *not* a reasonable worry about arithmetic, it seems that semantic physicalism is false.

Similar arguments can be run against semantic psychologism. For starters, it seems clear that if this view were right, then it would be reasonable to worry that there aren't enough mental objects in the universe to make our mathematical sentences and theories true. To appreciate this, imagine that after being taught Euclid's proof (or Cantor's proof), Big Red raised his hand and said this:

> There couldn't be infinitely many prime numbers (or infinitely many transfinite cardinals) because my psychology professor told me that there are only finitely many ideas in each human head, and my astronomy professor told me there are no aliens with thoughts, and so there are only finitely many mental objects in the whole universe.

Once again, it seems reasonable to conclude from this little speech that Big Red *doesn't understand*. In the context of Euclid's and Cantor's proofs, it doesn't *matter* how many mental objects there are in the universe. Even if it's true that there are only finitely many mental objects in the universe, this is simply not a good reason to reject the two proofs. And the only reasonable conclusion we can draw from this, I think, is that the two theorems—i.e., the sentences that say that there are infinitely many primes and infinitely many transfinite cardinals—should not be interpreted as being about actual mental objects that exist in our heads.

It's important to note that the worry here is not that humans can't *conceive* of an infinite set. The worry has to do with the number of actual mental objects (i.e., distinct number-ideas, or set-ideas) that are *actually residing* in human heads. Semantic psychologism implies that in order for standard arithmetical theories like Peano Arithmetic (PA) to be true, there has to be an infinite number of actual mental objects. Why? Because PA implies that there are infinitely many numbers; it implies that there is such a thing as the number 1, and there is such a thing as the number 2, and 2 is not identical to 1, and so on. Thus, if semantic psychologism were right, then the truth of PA would depend on there actually existing infinitely many distinct number-ideas in human heads. But, in fact, the truth of PA clearly *doesn't* depend on this; if you're worried that PA might be false because there aren't enough actual ideas to go around, then that just shows that you don't understand what PA *says*. And so the conclusion we should draw here is that semantic psychologism is false.

A second problem with semantic psychologism is that it implies that our mathematical theories are empirical theories and that the right methodology for determining whether there are, say, infinitely many primes would involve an empirical investigation into the number of actual number-ideas that exist in the universe. In other words, semantic psychologism implies that Euclid's proof is not just mistaken but completely wrongheaded in its methodology. If semantic psychologism were true, then Euclid should have used empirical methods. But, of course, this is crazy. There's nothing wrong with the method of mathematical proof; the problem here is that semantic psychologism is false.

You might complain that these arguments are directed at a silly or trivial version of semantic psychologism that no one would ever endorse. Well, I agree with that. The view is crazy. (As Frege says, "Weird and wonderful . . . are the results of taking seriously the suggestion that number is an idea."[6]) But the problem is that there's no way to get rid of the craziness, or the silliness, without altering the view in a way that makes it the case that it's no longer a psychologistic view at all. Suppose, for instance, that someone said something like this:

> The psychologistic view isn't that mathematics is about *actual* ideas that really exist inside of human heads. We can take the view to be that mathematical sentences are about what it's *possible* to do in our heads. For instance, to say that there are infinitely many prime numbers is not

to say that there really exists an actual infinity of prime-number ideas inside of human heads; it's to say that it's possible to construct infinitely many prime numbers in our heads.

There are a few problems with this view. In the present context, the main problem is that the view described here isn't a version of semantic psychologism at all, and so it's no defense against the above objections. Semantic psychologism is the view that mathematical sentences are about mental objects. The above view rejects this, and so it's not a version of semantic psychologism. Rather, it's a version of non-literalism, or non-Face-value-ism; in particular, on this view, "3 is prime" doesn't say that a certain object (namely, 3) is prime; rather, it says something about what it's possible for humans to do. But as an empirical hypothesis about what ordinary people actually mean when they utter sentences like "3 is prime," this is extremely implausible; there's simply no *evidence* that people really mean to say things like this when they utter sentences like "3 is prime." (Again, the only people who ever mean things like this by mathematical claims are people who are worried about philosophy.)

In any event, if we stick with semantic psychologism, the view is completely implausible for the reasons given above. And if this is right, then the only option we're left with is semantic platonism. Now, you might object here that just as there are reasons to resist semantic physicalism and semantic psychologism, so too there are reasons to resist semantic platonism. For you might think it's implausible that ordinary people intend to be speaking of abstract objects when they say things like "3 is prime." But semantic platonists don't need to say that people have such intentions, and indeed, they *shouldn't* say this. What they should say is that (a) people are best interpreted as speaking literally when they say things like "3 is prime," and so these sentences have to be taken as being about objects (in particular, numbers); and (b) our semantic intentions are straightforwardly incompatible with semantic physicalism and semantic psychologism, and so there's no way to interpret us as talking about physical or mental objects when we say things like "3 is prime" (this is what the above arguments show); and (c) there's nothing in our intentions that's incompatible with semantic platonism; and so (d) even if people don't have a positive intention to be referring to abstract objects when they say things like "3 is prime," the best interpretation of these utterances has it that they *are* about abstract objects—or at any rate that they're *supposed* to be about abstract objects, or that they *purport* to be about abstract objects, or some such thing.

So if all of the arguments in this section are correct, then semantic platonism is the best semantic theory of ordinary mathematical discourse. And it's worth noting that the argument I've given for this claim is entirely *empirical*. It's based on considerations about what actual people (mathematicians and ordinary folk) are really saying when they utter sentences like "3 is prime." In a nutshell, the idea is that (a) we should interpret people as

speaking literally when they engage in talk of numbers, and (b) there's no plausible way of taking them to be talking about physical or mental objects. It's hard to see how Mill could plausibly deny either of these claims. And so it seems to me that we have good reasons to endorse semantic platonism.

(In response to my claim that the argument I've constructed here is empirical, you might ask what the relevant empirical *data* are. The answer is that the data I'm using here are the intuitions of ordinary speakers—e.g., the intuition that facts about how many physical objects there are in the universe are completely irrelevant to the truth values of ordinary mathematical claims about how many prime numbers and transfinite cardinals there are. Now, I haven't actually performed an "X-phi" study to verify that people really do have this intuition; but I think it's pretty obvious that anyone who understood the relevant sentences (i.e., sentences like "There are infinitely many prime numbers" and "There are infinitely many transfinite cardinals") would have this intuition. Maybe I'm wrong about this, but if I'm right, then I think we have good empirical reason to reject the idea that these sentences should be interpreted as being about physical objects.)

III.ii. Fictionalism

It might seem that semantic platonism is incompatible with Mill's background philosophical commitments. Indeed, it seems likely to me that something like this thought is why Mill jumped ship at the start and denied that arithmetical sentences like "3 is prime" should be read at face value. For insofar as Mill wants to endorse a physicalistic ontology, he can't very well countenance the existence of non-physical abstract objects. But I have not argued here for the existence of abstract objects. All I've argued for is a *semantic* conclusion. What I've argued, in a nutshell, is that our mathematical sentences and theories are *supposed* to be about abstract objects—or that we should read them as *purporting* to be about abstract objects, or some such thing. But it doesn't follow from this that there *are* abstract objects. To help bring this point out, let's switch to a different case. Consider the following semantic theory:

> *Semantic theism*: The word "God" (in ordinary English) is supposed to refer to an intelligent creator of the universe—i.e., it should be interpreted as *purporting* to refer to such a creature.

It seems to me that this theory is obviously true. But, of course, it doesn't follow from this that *theism* is true, i.e., that there really is a God who created the universe. And the same thing is true in the case of mathematics. From the mere fact that we use the numeral "3" in a way that makes it the case that it's supposed to refer to an abstract object (namely, the number 3), it doesn't follow that there really is such a thing as the number 3.

But whatever we say about theism and semantic theism, it might seem that in the case of mathematics, there is an easy argument from semantic platonism to platonism. In particular, it might seem that we can argue in the following way:

(i) Semantic platonism is true—i.e., ordinary mathematical sentences like "3 is prime" are straightforward claims about abstract objects (or more precisely, they're *supposed* to be about abstract objects). Therefore,
(ii) Mathematical sentences like "3 is prime" could be true only if platonism were true—i.e., only if abstract objects really existed. But
(iii) Mathematical sentences like "3 is prime" *are* true. Therefore,
(iv) Platonism is true (where platonism is just the view that there exist abstract objects and our mathematical theories are descriptions of these objects).

Prima facie, this argument seems very compelling. I've already argued for premise (i). Moreover, (ii) seems to follow immediately from (i). To appreciate this point, think of the sentence "Mars is red"; this sentence couldn't be true unless Mars existed. And likewise, given (i), "3 is prime" couldn't be true unless an abstract object existed; in particular, it couldn't be true unless the number 3 existed. So it seems that (ii) is true. Moreover, (iii) seems entirely obvious, and when we put (ii) and (iii) together, they entail platonism. So it seems that semantic platonism leads directly to platonism.

But there's a way out of this argument. We can reject premise (iii) and endorse mathematical fictionalism. This view can be defined as follows:

> *Mathematical fictionalism:* (a) semantic platonism is true—i.e., ordinary mathematical sentences like "3 is prime" are supposed to be about abstract objects (or they *purport* to be about abstract objects, or some such thing); but (b) there are no such things as abstract objects; and so (c) ordinary mathematical sentences like "3 is prime" are not true.[7]

Fictionalism doesn't just give us a way of responding to the above argument for platonism. It is also, I think, capable of giving Mill everything he wants in a philosophy of mathematics. But to appreciate this point, we first need to develop the view in a bit more detail. There are a number of objections that you might raise against fictionalism, but the most obvious one is probably the following:

> Since fictionalism entails that sentences like "3 is prime" are untrue, it seems to give us no account of the difference between "3 is prime" and, say, "3 is even." It seems beyond doubt that there's some important sense in which "3 is prime" is *right*, or *correct*, or some such thing, whereas "3 is even" is *not* correct. This seems to be an objective fact that we can't simply ignore, and so fictionalists need to say what the correctness of sentences like "3 is prime" consists in.

Hartry Field (1989) responded to this worry by claiming that the sense in which "3 is prime" is correct and "3 is even" is incorrect is roughly equivalent to the sense in which "Alice entered Wonderland by falling down a rabbit hole" is correct and "Alice entered Wonderland by falling down a man hole" is incorrect. More specifically, on Field's view, the so-called correctness of "3 is prime" consists in the fact that it's *true in the story of mathematics*, or *part of* the story of mathematics.

This, I think, is a good start, but fictionalists need to say more. In particular, they need to say what the story of mathematics consists in. According to Field (1998), the story of mathematics consists (roughly) in the various axiom systems that are accepted in the various branches of mathematics, and so on his view, the relevant sort of mathematical correctness—what we might call "fictionalistic mathematical correctness"—comes down (roughly) to following from accepted axioms. But I have argued elsewhere (2009) that this view can't be right and that fictionalists should instead endorse the view that the story of mathematics consists in the claim that platonism is true, i.e., the claim that there really do exist abstract objects of the kinds that our mathematical theories purport to be about. On this view, we can say that a mathematical sentence S is *fictionalistically correct* if and only if it would have been true if platonism had been true; or what is perhaps simpler, we can say that S is fictionalistically correct if and only if the following sentence is strictly and literally true:

> (S#) Necessarily, if platonism is true (i.e., if there exist abstract objects of the kinds that our mathematical theories purport to be about), then S is true.

If we endorse this view, then all the mathematical sentences that we ordinarily think of as true (e.g., "3 is prime") will be fictionalistically correct, and all the mathematical sentences that we ordinarily think of false (e.g., "3 is even") will *not* be fictionalistically correct. And given this, fictionalists can say that what really matters in mathematics is fictionalistic correctness, not literal truth. (I think it can also be argued that fictionalistic correctness is what matters in the applications of mathematics, i.e., in the use that empirical scientists make of mathematics; I can't argue this point here, but see my (1998).)

There's a lot more that could be said in defense of mathematical fictionalism, but I think I've said enough to motivate the claim that this view fits perfectly with Mill's overall philosophical view. First, it is clearly a naturalistic, physicalistic view; i.e., it doesn't commit to the existence of any metaphysically occult objects. Second, fictionalism is perfectly compatible with a thoroughgoing empiricism. The only sentences in the neighborhood of mathematics that are literally (and non-vacuously[8]) true, on this view, are sentences of the form (S#), e.g., sentences like "Necessarily, if platonism is true (i.e., if there exist abstract objects of the kinds that we have in mind when we do mathematics), then 3 is prime." But sentences like this are *analytic*; their antecedents analytically entail their consequents, and so knowledge of these sentences should be compatible with empiricism. Indeed, Mill is already committed to the idea that we can have knowledge of such sentences. This came out in our discussion of Mill's view of geometry; in particular, it is very clear in the following passage from Mill:

> When, therefore, it is affirmed that the conclusions of geometry are necessary truths, the necessity consists in reality only in this, that they correctly follow from the suppositions from which they are deduced.[9]

Given this stance of Mill's, I think he ought to allow that sentences of the form (S#) are true and that we can have knowledge of these sentences by simply working out the consequences of the existence of things like the natural-number sequence. In other words, I think Mill ought to see the fictionalist line here as a plausible alternative to his own view.

(Mill might want to resist fictionalism because he might think that platonism is *impossible*, and given this, he might say that *every* mathematical sentence comes out as fictionalistically correct. I think this response is misguided, but I can't get into this here. It's worth noting, however, that this stance isn't forced on us by the other background philosophical views that I've been assuming in this chapter; i.e., it's not forced on us by endorsing views like naturalism, anti-platonism, empiricism and so on.)

Now, of course, if we endorse fictionalism, then we'll have to say that ordinary mathematical sentences like "3 is prime" are strictly speaking false (or at least untrue), and you might think that this is a cost. To this, I have two things to say. First, it's hard to see why *Mill* would consider it a cost. We've already seen that he was perfectly comfortable with the idea that certain kinds of mathematical claims that we ordinarily think of as obviously true are, in fact, *false*. Once again, this emerged in our discussion of Mill's philosophy of geometry. The claim that all of the radii of a circle are exactly equal in length is, according to Mill, strictly speaking false. On Mill's view, there are no such things as perfect circles; the only circles that really exist are imperfect physical circles; but in connection with these imperfect circles, it's simply not true that all of their radii are all exactly equal in length. So, again, Mill was already committed to the idea that certain mathematical sentences that seem obviously true are actually false. And given this, it's hard to see why he would balk at the claim that "3 is prime" is false—i.e., at the claim that whereas "3 is prime" is obviously "right" in some sense of the term (in particular, whereas it's fictionalistically correct), it isn't strictly speaking true.

The second point I want to make in response to the above worry is just this: I think it can be argued that there actually isn't any substantive cost to claiming that ordinary mathematical sentences like "3 is prime" are strictly

and literally false. For I think it can be argued that what really matters in mathematics is fictionalistic correctness, not literal truth, and I think it can also be argued that fictionalistic correctness gives us everything we might have rationally wanted out of mathematical truth. I obviously can't argue for these sweeping claims here, but see my (1998) and (2009).

In any event, if I'm right about this, then there aren't any costs to Mill for endorsing mathematical fictionalism. But there is a huge gain. For unlike Mill's own view, fictionalism dovetails with the actual facts about mathematical practice. It doesn't have any of the odd consequences that Mill's view does. In particular, fictionalism doesn't entail the cogency of the weird worries described above about the proofs of Euclid and Cantor. If fictionalism is the right philosophy of mathematics, then when the above-described students object to the proofs of Euclid and Cantor on the grounds that there aren't enough physical objects in the universe to make their theorems true, we can respond in the way that we intuitively *want* to respond, i.e., by saying something like this:

> What are you talking about? The theorems of Euclid and Cantor aren't about physical objects. They're about mathematical objects. Or at any rate, they *purport* to be about mathematical objects; in particular, they purport to be about numbers and sets.

And if someone responded to this by pointing out that there are no such things as mathematical objects like numbers and sets, then we could respond by saying this: "So what? All that shows is that the theorems of Euclid and Cantor aren't literally *true*. But that doesn't matter. They're still *good* in the sense that matters in mathematics because they're fictionalistically correct. In other words, they're true in the story of mathematics. Or to put the point still another way, the theorems of Euclid and Cantor are good, or fictionalistically correct, because the following claim is strictly and literally true: necessarily, if there are abstract objects of the kinds that platonists have in mind (i.e., the kinds that our mathematical theories purport to be about), then there are infinitely many prime numbers, and there are infinitely many transfinite cardinals that keep getting bigger and bigger without end.

In short, then, my conclusion is that Mill should have endorsed mathematical fictionalism. If he had done this, he could have hung onto views like naturalism, anti-platonism, empiricism and so on while providing a much more satisfying and plausible theory of mathematics.

NOTES

1. Mill (1843, *CW*: VII.226). Emphasis added. All references to Mill's works are to the authoritative edition of *Collected Works of John Stuart Mill* (Mill, 1963–1991—cited as *CW*, followed by volume and page number), unless otherwise indicated.
2. Mill (1843, *CW*: VII.227).

3. Mill uses the term "collections", but it's pretty clear that he's *not* thinking of *sets*; or at any rate, he's not thinking of sets in the way that we think of them today; in particular, he's not thinking of collections as abstract objects. He's thinking about purely physical collections. I will talk of these as *piles* to make their physicality clear, but of course I do not mean to suggest by this that the objects have to be piled on top of each other. Thus, e.g., on my lingo, the Eifel Tower and the Empire State Building form a perfectly good "pile."
4. Mill (1843, *CW*: VII.254–55).
5. Mill (1843, *CW*: VII.254–55).
6. Frege (1884: section 27).
7. This view was first introduced by Field (1980), and it has been further developed by myself (1998), Rosen (2001), Yablo (2002) and Leng (2010).
8. According to fictionalists, lots of universal mathematical sentences are vacuously true. E.g., "All even numbers are divisible by 2" is true for the simple reason that there *are* no numbers; i.e., it's true for the same reason that "All unicorns are purple" is true. But this is completely unimportant and uninteresting. After all, "All even numbers are *purple*" comes out true on this view as well.
9. Mill (1843, *CW*: VII.227).

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