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MATH 540B
Due: 3/5/08

HW#5

$\boxed{13.6}$ 3) [Prove that if a field contains the n^{th} roots of unity for n odd then it also contains the $2n^{th}$ roots of unity.]

Pf: Let F_n be a field containing the n^{th} roots of unity for n odd. Then $\xi_n \in F_n$. Let's assume that $\mathbb{Q} \subseteq F_n$. So, $\mathbb{Q}(\xi_n) \subseteq F_n$. By Corollary 42, $[\mathbb{Q}(\xi_n):\mathbb{Q}] = \phi(n)$.

Now consider $\mathbb{Q}(\xi_{2n})$. By Corollary 42, $[\mathbb{Q}(\xi_{2n}):\mathbb{Q}] = \phi(2n)$. But since n is odd, $(2,n)=1$, so $\phi(2n) = \phi(2)\phi(n) = \phi(n)$. So, $[\mathbb{Q}(\xi_{2n}):\mathbb{Q}] = \phi(n)$. Now we have the following picture:

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\xi_{2n}) \arrow[dl, dash, "1"'] \arrow[dr, dash] & \\
\mathbb{Q}(\xi_{2n}) \arrow[dr, dash, "\phi(n)"'] & & \mathbb{Q}(\xi_n) \arrow[dl, dash, "\phi(n)"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Note that $\mathbb{Q}(\xi_n) \subseteq \mathbb{Q}(\xi_{2n})$ since $\mu_n \subseteq \mu_{2n}$ (since $n | 2n$). But the diagram necessitates $[\mathbb{Q}(\xi_{2n}):\mathbb{Q}(\xi_n)] = 1$, so in fact $\mathbb{Q}(\xi_{2n}) = \mathbb{Q}(\xi_n)$!

So, $\mathbb{Q}(\xi_{2n}) = \mathbb{Q}(\xi_n) \subseteq F_n$, so F_n contains the $2n^{th}$ roots of unity. ▨

Alternate, more general proof:
Suppose F is a field that contains the n^{th} roots of unity. Then $\xi_n^a \in F$ for $0 \le a \le n-1$. Consider $x^{2n}-1 = (x^n-1)(x^n+1)$. The $2n^{th}$ roots of unity will be the n^{th} roots of unity, and the roots of x^n+1. What are the roots of x^n+1? They are simply $-\xi_n^a$ for $0 \le a \le n-1$, where $-\xi_n^a \in F$. This works since n is odd, so $(-\xi_n^a)^n + 1 = -(\xi_n^a)^n + 1 = -((\xi_n^a)^n - 1) = 0$. Thus all the roots of $x^{2n}-1$ are in F. ▨

DB 2

$\boxed{14.1}$ 5) [Determine the automorphisms of the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ explicitly.]

First off, we know that $|Aut(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2}))| \leq [\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}(\sqrt{2})]$, so let's find $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}(\sqrt{2})]$. Consider $x^2 - \sqrt{2} \in \mathbb{Q}(\sqrt{2})[x]$. This has $\sqrt[4]{2}$ as a root. Note also that $m_{\sqrt[4]{2}, \mathbb{Q}(\sqrt{2})}$ cannot have degree 1, since $\sqrt[4]{2} \notin \mathbb{Q}(\sqrt{2})$. So, $m_{\sqrt[4]{2}, \mathbb{Q}(\sqrt{2})} = x^2 - \sqrt{2}$. Thus, $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}(\sqrt{2})] = 2$.

So, we have either 1 or 2 automorphisms. Let $\tau \in Aut(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2}))$. Then τ sends $\sqrt[4]{2}$ to a root of $x^2 - \sqrt{2}$, that is, to $\sqrt[4]{2}$ or $-\sqrt[4]{2}$. So, we have 2 automorphisms:

$id : \sqrt[4]{2} \longmapsto \sqrt[4]{2}$

$\tau : \sqrt[4]{2} \longmapsto -\sqrt[4]{2}$

DB3

[Find the Galois group of x^4-2 over $\mathbb{Q}$. Show that it is not abelian.]

Since x^4-2 is separable, this makes sense. So, the Galois group of x^4-2 over $\mathbb{Q}$ is $\mathrm{Aut}(E/\mathbb{Q})$, where $E=\mathbb{Q}(\sqrt[4]{2},-\sqrt[4]{2},\sqrt[4]{2}i,-\sqrt[4]{2}i)$, the splitting field of x^4-2 over $\mathbb{Q}$. By Prop 5 (p. 562) and our HW, $|\mathrm{Aut}(E/\mathbb{Q})| \underset{\text{Prop 5}}{=} [E/\mathbb{Q}] \underset{\text{HW}}{=} 8$. So, we're looking for 8 automorphisms.

Also by HW, note that $E=\mathbb{Q}(\sqrt[4]{2},i)$. By Prop 2, we must have that any automorphism sends $\sqrt[4]{2}$ to $\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i$, or $-\sqrt[4]{2}i$, (roots of minimal polynomial of $\sqrt[4]{2}$, x^4-2) and sends i to i or $-i$. (roots of minimal polynomial of i, x^2+1) This gives us 8 choices, and hence our 8 automorphisms. Listed, they are:

$$1:\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2}\\ i\mapsto i\end{cases}\qquad \sigma_1:\begin{cases}\sqrt[4]{2}\mapsto-\sqrt[4]{2}\\ i\mapsto i\end{cases}$$

$$\sigma_2:\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2}i\\ i\mapsto i\end{cases}\qquad \sigma_3:\begin{cases}\sqrt[4]{2}\mapsto-\sqrt[4]{2}i\\ i\mapsto i\end{cases}$$

$$\sigma_4:\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2}\\ i\mapsto -i\end{cases}\qquad \sigma_5:\begin{cases}\sqrt[4]{2}\mapsto-\sqrt[4]{2}\\ i\mapsto -i\end{cases}$$

$$\sigma_6:\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2}i\\ i\mapsto -i\end{cases}\qquad \sigma_7:\begin{cases}\sqrt[4]{2}\mapsto-\sqrt[4]{2}i\\ i\mapsto -i\end{cases}$$

$\mathrm{Aut}(E/\mathbb{Q})$ is not abelian since $\sigma_6\sigma_5\neq\sigma_5\sigma_6$, as is shown below:

$$(\sigma_6\sigma_5)(\sqrt[4]{2})=\sigma_6(-\sqrt[4]{2})=-\sigma_6(\sqrt[4]{2})=-\sqrt[4]{2}i$$

not equal!

$$(\sigma_5\sigma_6)(\sqrt[4]{2})=\sigma_5(\sqrt[4]{2}i)=\sigma_5(\sqrt[4]{2})\sigma_5(i)=(-\sqrt[4]{2})(-i)=\sqrt[4]{2}i$$