4680 - HW 2
Solutions

1(a) $e^{2+i} = e^2 e^i = e^2 e^{1\cdot i}$

$= e^2 \left[\cos(1) + i\sin(1)\right]$

$= e^2\cos(1) + i\, e^2 \sin(1)$

1(b) $\sin(2-i) = \dfrac{e^{i(2-i)} - e^{-i(2-i)}}{2i}$

$= \dfrac{e^{1+2i} - e^{-1-2i}}{2i} = \dfrac{e^1\left[\cos(2)+i\sin(2)\right] - e^{-1}\left[\cos(-2)+i\sin(-2)\right]}{2i}$

$= \dfrac{-i}{2}\left[e\cos(2) - \frac{1}{e}\cos(2) + i\, e\sin(2) + i\frac{1}{e}\sin(2)\right]$

$\dfrac{1}{2i} = \dfrac{1}{2i}\cdot\dfrac{i}{i} = \dfrac{i}{-2}$

$= \left[\frac{1}{2}e\sin(2) + \frac{1}{2e}\sin(2)\right]$

$+ i\left[-\frac{e}{2}\cos(2) + \frac{1}{2e}\cos(2)\right]$

$\cos(-2) = \cos(2)$

$\sin(-2) = -\sin(2)$

1(c) $e^{3-\pi i} = e^{3} e^{-\pi i}$

$= e^{3}\left[\cos(-\pi) + i\sin(-\pi)\right]$

$= e^{3}\left[-1 + 0i\right] = -e^{3}$

1(d) $\cos(3\pi + i) = \dfrac{e^{i(3\pi+i)} + e^{-i(3\pi+i)}}{2}$

$= \frac{1}{2}\left[e^{3\pi i} e^{-1} + e^{-3\pi i} e^{1}\right]$

$= \frac{1}{2}\left[\frac{1}{e}\left[\cos(3\pi) + i\sin(3\pi)\right] + e\left(\cos(-3\pi) + i\sin(-3\pi)\right)\right]$

$= \frac{1}{2}\left[\frac{1}{e}\left[-1 + 0i\right] + e\left[-1 + 0i\right]\right]$

$= \frac{1}{2}\left[-\frac{1}{e} - e\right] = -\dfrac{e + \frac{1}{e}}{2}$

2 (a)

$$\log(1) = \ln|1| + i\arg(1)$$
$$= 0 + i(2\pi k), \quad k \in \mathbb{Z}$$

If we choose the branch $(0, 2\pi)$ or $[-\pi, \pi)$ we get $\arg(1) = 0$.

So in either of those cases, $\log(1) = 0$

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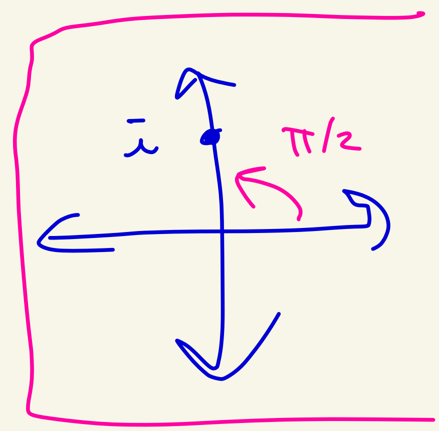
2 (b) $$\log(i) = \ln|i| + i\arg(i)$$
$$= 0 + i\left[\frac{\pi}{2} + 2\pi k\right] \quad k \in \mathbb{Z}$$



branch $[0, 2\pi)$: $\arg(i) = \frac{\pi}{2}$

branch $[-\pi, \pi)$: $\arg(i) = \frac{\pi}{2}$

So in either case,

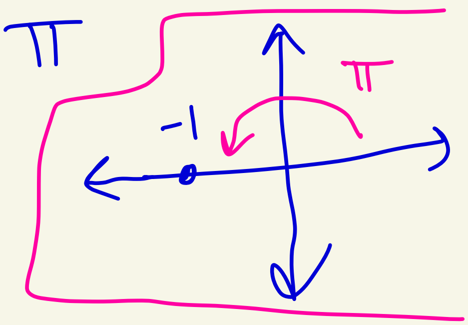
$$\log(i) = 0 + i\frac{\pi}{2} = \frac{\pi}{2}i$$

2(c)

$$\log(-1) = \ln|-1| + i\arg(-1)$$

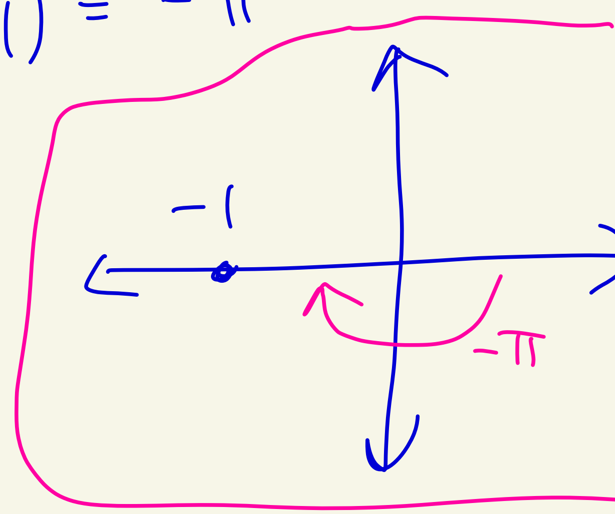
$$= \underbrace{\ln(1)}_{0} + i(\pi + 2\pi k), \quad k \in \mathbb{Z}$$

$[0, 2\pi)$ branch: $\arg(-1) = \pi$



$$\log(-1) = \pi i$$

$[-\pi, \pi)$ branch: $\arg(-1) = -\pi$



$$\log(-1) = -\pi i$$

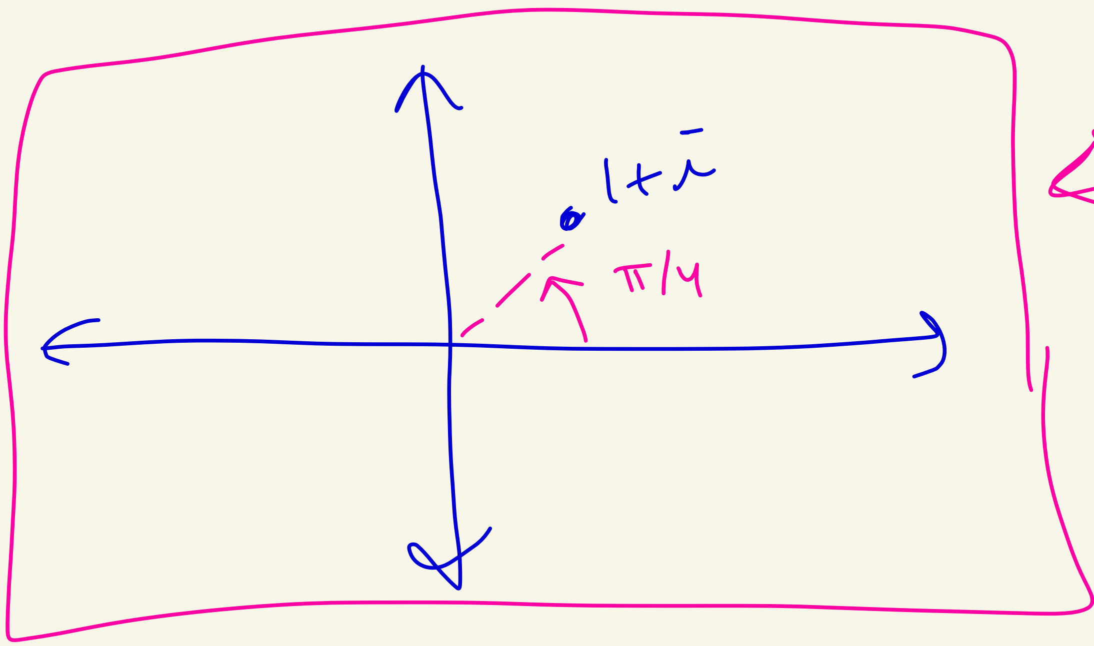
2(d)

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi k\right), \; k \in \mathbb{Z}$$

$[0, 2\pi)$ or $[-\pi, \pi)$ branch:

$$\arg(1+i) = \frac{\pi}{4}$$

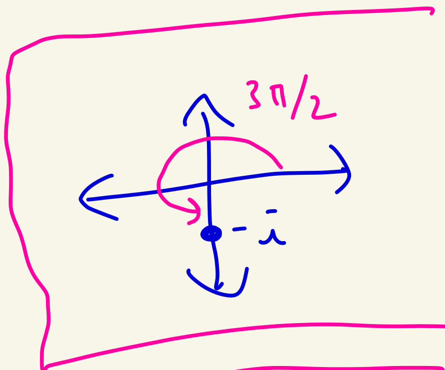
$$\log(1+i) = \ln(\sqrt{2}) + i\frac{\pi}{4}$$



2(e)

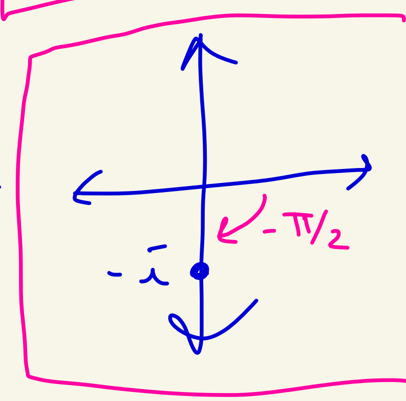
$$(-i)^i = e^{i\log(-i)}$$
$$= e^{i[\ln|-i| + i\arg(-i)]}$$
$$= e^{i[0 + i\arg(-i)]} = e^{-\arg(-i)}$$
$$= e^{-[\frac{3\pi}{2} + 2\pi k]}, \; k \in \mathbb{Z}$$

<u>$[0, 2\pi)$ branch:</u> $\arg(-i) = \frac{3\pi}{2}$



So, $(-i)^i = e^{-\frac{3\pi}{2}}$

<u>$[-\pi, \pi)$ branch:</u> $\arg(-i) = -\frac{\pi}{2}$



So, $(-i)^i = e^{\pi/2}$

2(f)

$$(-1)^i = e^{i\log(-1)} \overset{2(c)}{=} e^{i[i(\pi+2\pi k)]},\ k \in \mathbb{Z}$$

<u>$[0,2\pi)$ branch:</u> In this case, from 2(c) we get $\log(-1) = \pi i$ and so $(-1)^i = e^{i(\pi i)} = e^{-\pi}$

<u>$[-\pi,\pi)$ branch:</u> In this case, from 2(c) we get $\log(-1) = -\pi i$ and so, $(-1)^i = e^{i(-\pi i)} = e^{\pi}$

2(g)

$$2^{i} = e^{i\log(2)} = e^{i[\ln|2| + i\arg(2)]}$$

$$= e^{i[\ln(2) + i(0+2\pi k)]}$$

$$= e^{(i\ln(2) - 2\pi k)}, \quad k \in \mathbb{Z}$$

For both the $[0, 2\pi)$ and $[-\pi, \pi)$ branches we get $\arg(2) = 0$.

So, for these branches we get

$$2^{i} = e^{i\ln(2)}$$

3(a) $e^z = -1$

$$z = \log(-1) = \ln|-1| + i\arg(-1)$$

$$= 0 + i[\pi + 2\pi k], \quad k \in \mathbb{Z}$$

$$= i[\pi + 2\pi k], \quad k \in \mathbb{Z}$$

3(b) $e^z = 1 + i\sqrt{3}$

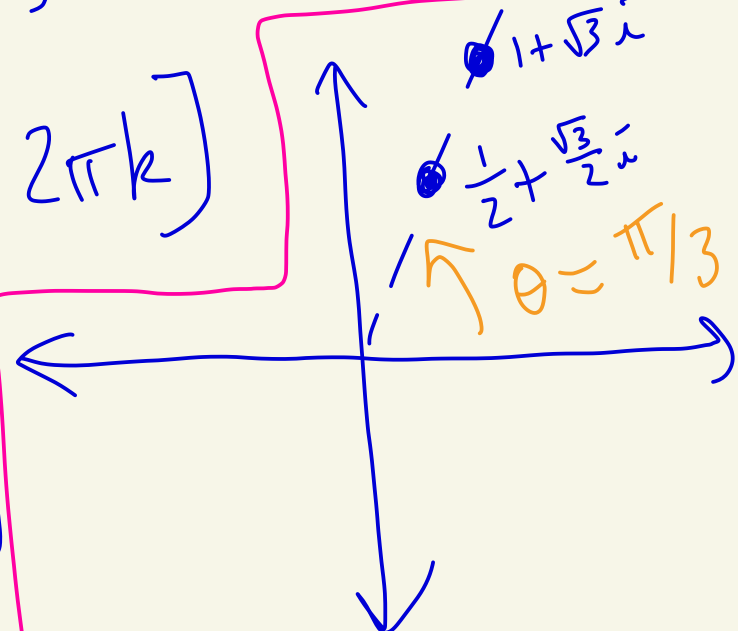
$$z = \log(1 + i\sqrt{3}) = \ln|1 + i\sqrt{3}| + i\arg(1 + i\sqrt{3})$$

$$= \ln\left(\sqrt{1^2 + \sqrt{3}^2}\right) + i\left[\frac{\pi}{3} + 2\pi k\right]$$

$k \in \mathbb{Z}$

$$= \ln(2) + i\left[\frac{\pi}{3} + 2\pi k\right]$$

$k \in \mathbb{Z}$



3 (c)

$$\cos(z) = 4 \iff \frac{e^{iz} + e^{-iz}}{2} = 4$$

$$\iff e^{iz} + e^{-iz} - 8 = 0$$

Multiply the equation by e^{iz} to get:

$$e^{2iz} + 1 - 8e^{iz} = 0$$

OR $$(e^{iz})^2 - 8e^{iz} + 1 = 0$$

that is $$w^2 - 8w + 1 = 0$$

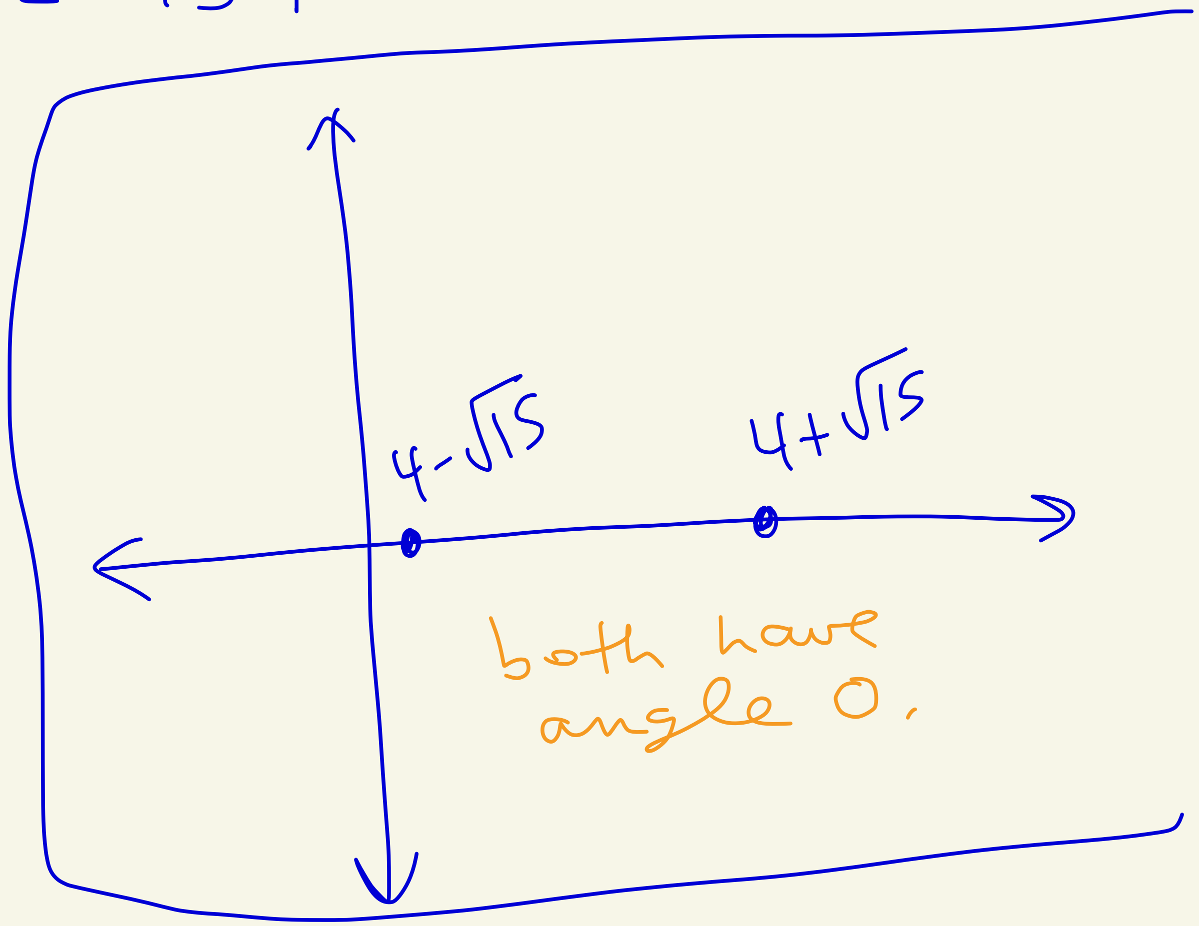
Using the quadratic formula we get

$$w = e^{iz} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 1}}{2}$$

$$= \frac{8 \pm \sqrt{60}}{2} = 4 \pm \sqrt{15}$$

So, $e^{iz} = 4 \pm \sqrt{15}$

Thus, $iz = \log|4 \pm \sqrt{15}|$.



Note $\frac{1}{i} = -i$.

So,

$$z = -i \log|4 \pm \sqrt{15}|,$$

$$z = -i\left[\ln(4 \pm \sqrt{15}) + i(0 + 2\pi k)\right]$$

$$= 2\pi k - i\ln(4 \pm \sqrt{15}), \quad k \in \mathbb{Z}$$

3(d)

$$\sin(z) = 4$$

$$\frac{e^{iz} - e^{-iz}}{2i} = 4$$

$$e^{iz} - e^{-iz} = 8i$$

multiply by e^{iz} to get

$$e^{2iz} - 1 = 8i\, e^{iz}$$

or, $(e^{iz})^2 - 8i(e^{iz}) - 1 = 0$

Giving $w^2 - 8i - 1 = 0$

where $w = e^{iz}$.

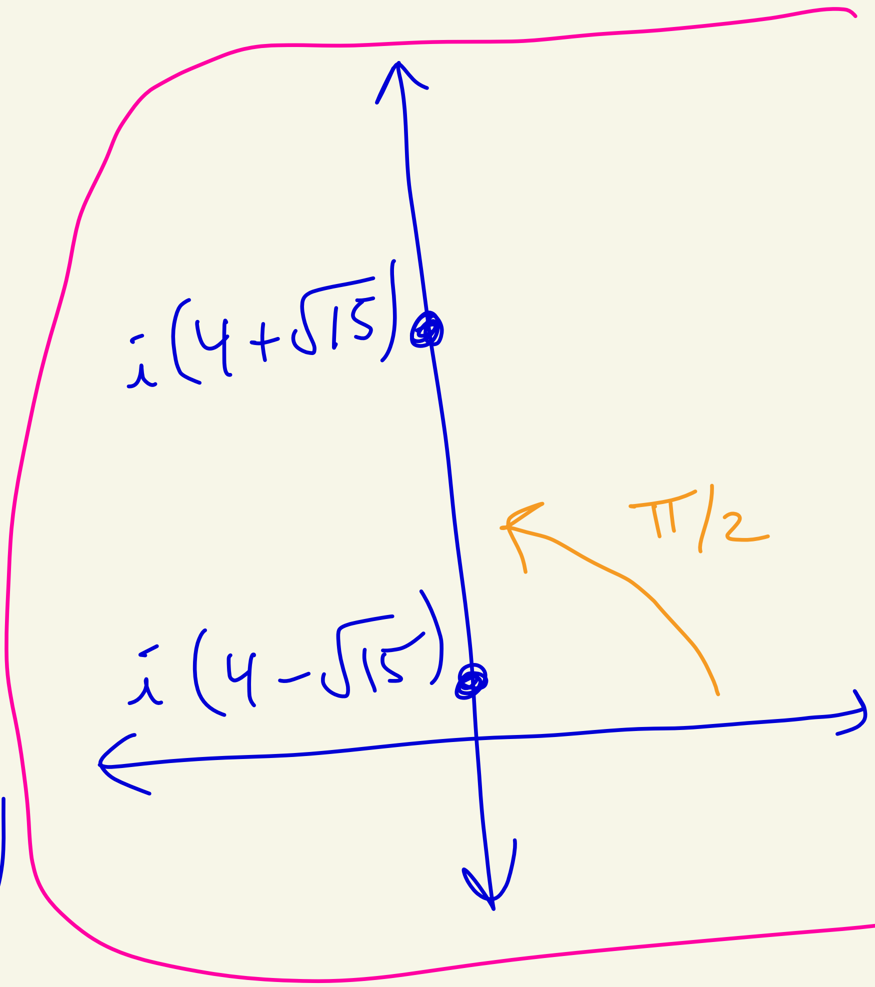
So, $w = e^{iz} = \dfrac{-(-8i) \pm \sqrt{(-8i)^2 - 4(-1)}}{2(1)}$

$$= \frac{8i \pm \sqrt{-64+4}}{2} = \frac{8i \pm \sqrt{-60}}{2}$$

$$= \frac{8i \pm i\sqrt{60}}{2} = 4i \pm i\sqrt{15}$$

$$= i\left(4 \pm \sqrt{15}\right)$$

$$e^{iz} = i\left(4 \pm \sqrt{15}\right)$$



$$iz = \log\left(i\left(4 \pm \sqrt{15}\right)\right)$$

multiply by $-i$ to get:

$$z = -i \log\left(i\left(4 \pm \sqrt{15}\right)\right)$$

$$z = -i\left[\ln\left|i\left(4 \pm \sqrt{15}\right)\right| + i\left(\frac{\pi}{2} + 2\pi k\right)\right]$$

$$= -i\ln\left(|i|\,|4 \pm \sqrt{15}|\right) + \frac{\pi}{2} + 2\pi k$$

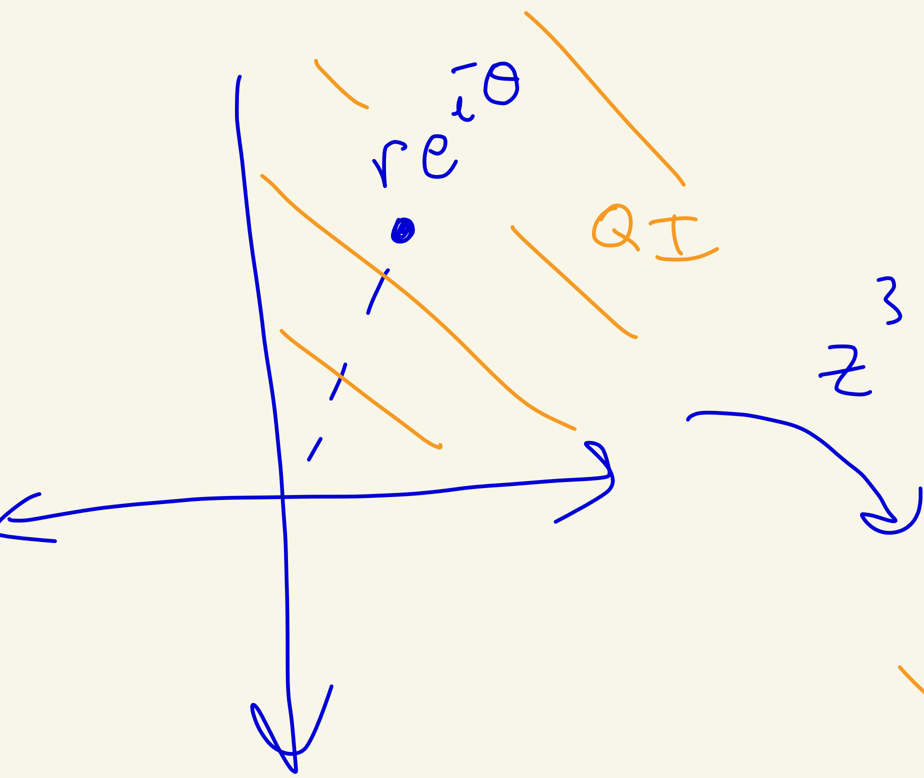
$$= -i\ln\left(4 \pm \sqrt{15}\right) + \frac{\pi}{2} + 2\pi k$$

$$= \left(\frac{\pi}{2} + 2\pi k\right) - i\ln\left(4 \pm \sqrt{15}\right)$$

$$k \in \mathbb{Z}$$

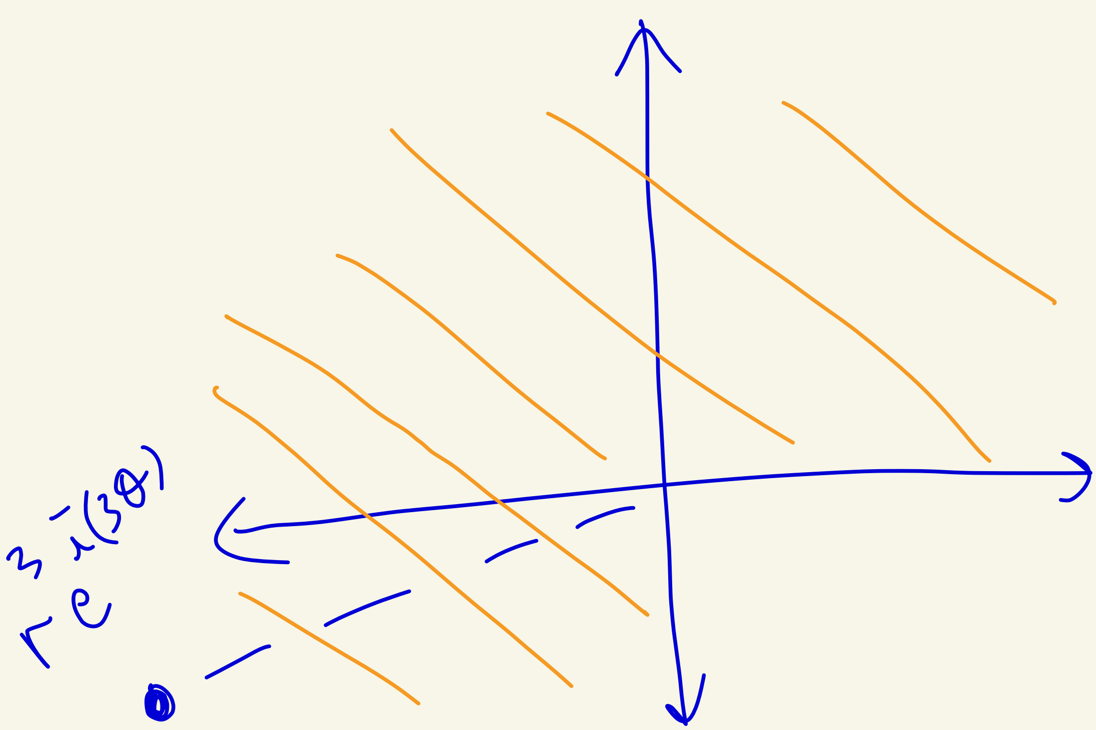
(4) Note that if $z = re^{i\theta}$ then $z^3 = r^3 e^{i(3\theta)}$

So, the length is cubed and the angle is tripled.



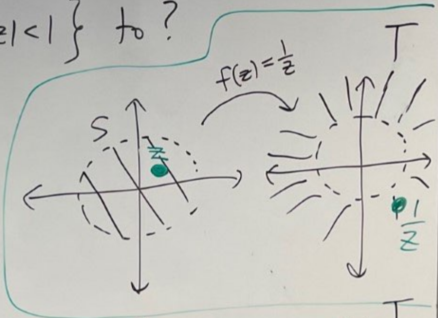
Answer:

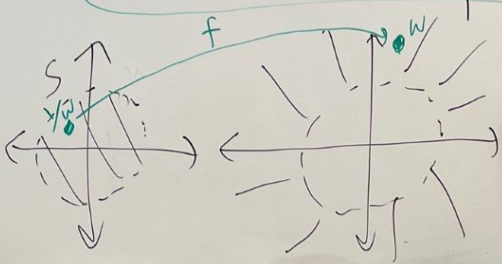
QI, QII, QIII



HW 2 #5

What does $f(z)=\frac{1}{z}$ map $S=\{z \mid |z|<1\}$ to?





Suppose $|z|<1$.

Then, $\left|\frac{1}{z}\right|=\frac{|1|}{|z|}=\frac{1}{|z|}>1$.

Let $T=\{w \mid |w|>1\}$.

We just showed $f(S)\subseteq T$.

Let's show $f(S)=T$.

Let $w\in T$. Then, $|w|>1$.

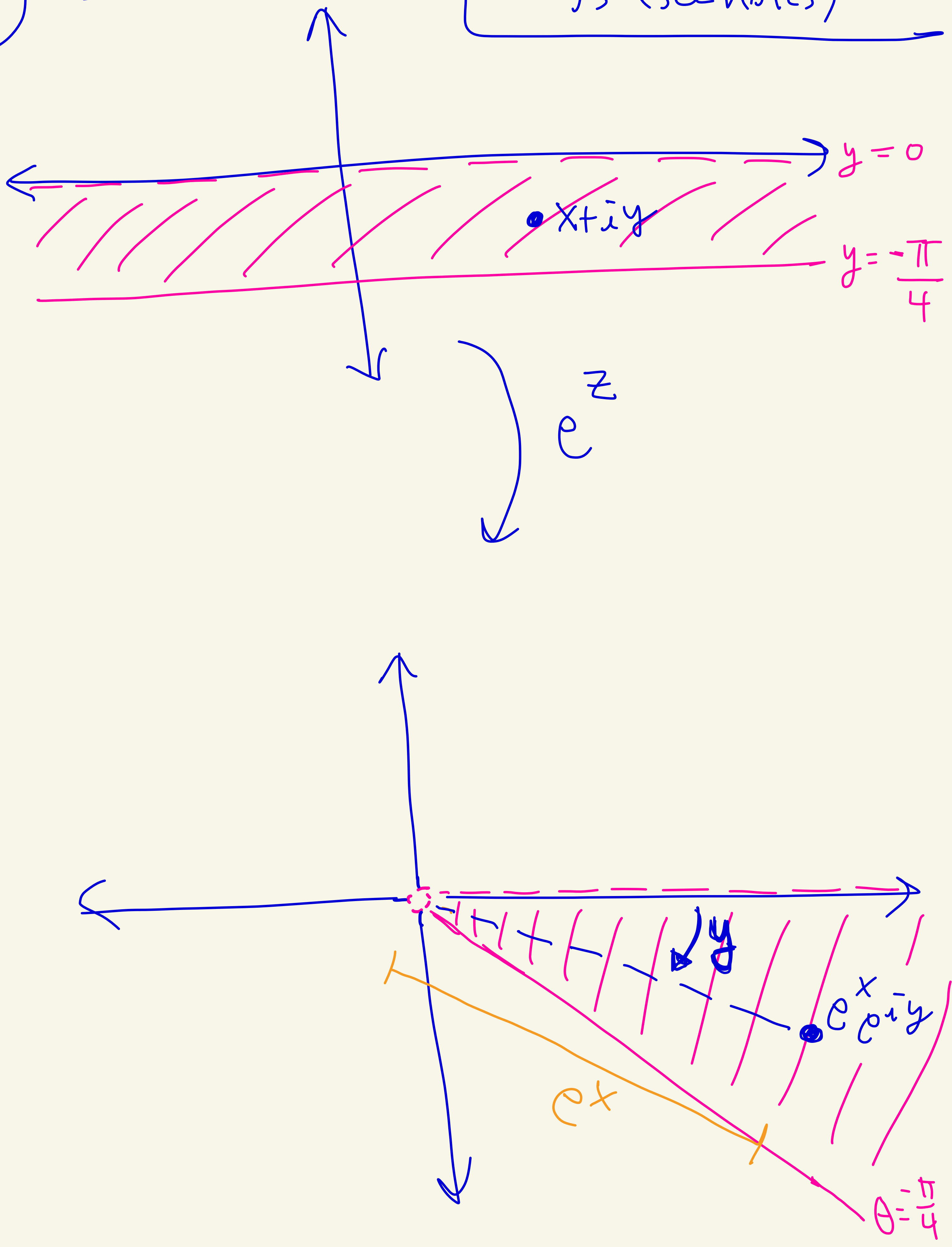
Then, $\frac{1}{|w|}<1$. So, $\left|\frac{1}{w}\right|<1$.

Thus, $\frac{1}{w}\in S$ and $f\left(\frac{1}{w}\right)=\frac{1}{\left(\frac{1}{w}\right)}=w$.

So, $f(S)=T$.

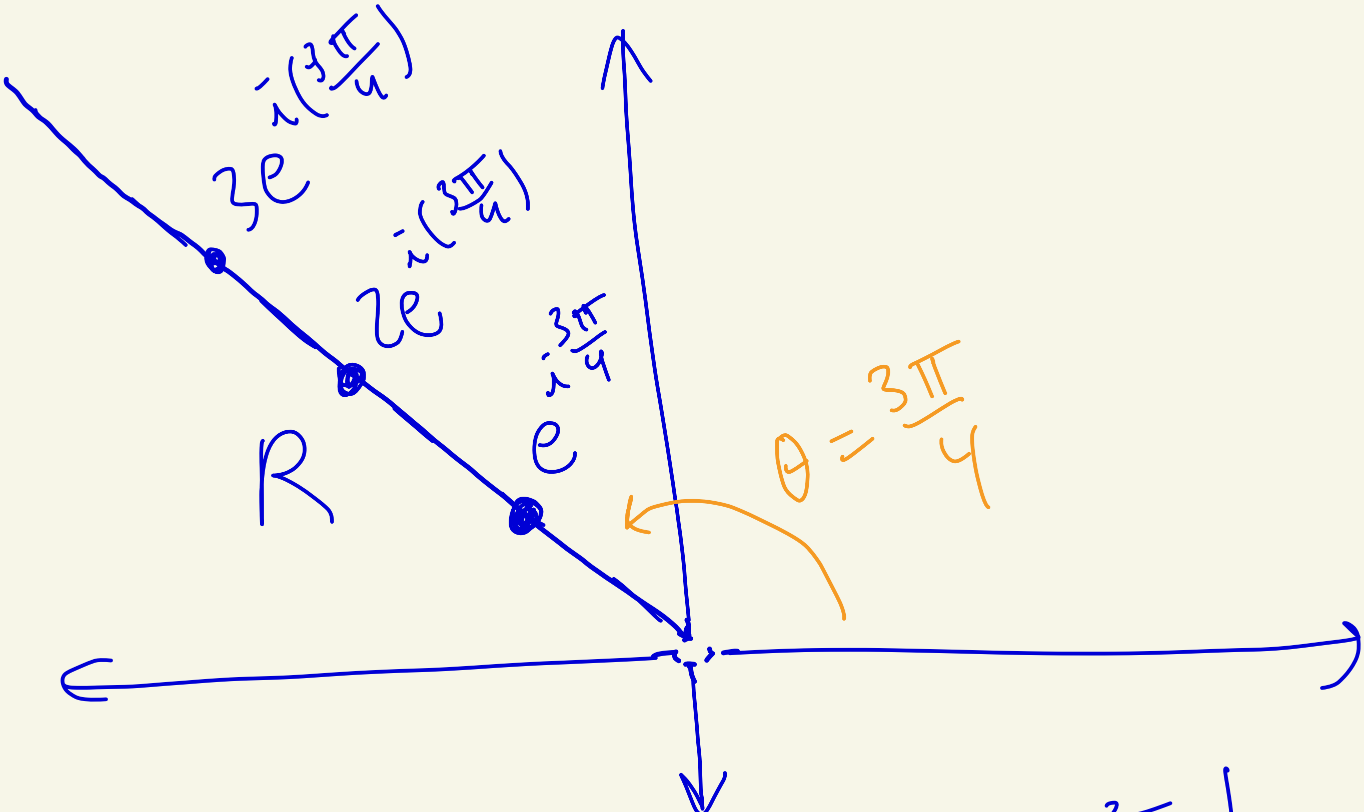
6

Recall that e^z sends horizontal lines to rays (see notes)



(7) $R = \{ re^{i(3\pi/4)} \mid r \in \mathbb{Z}, r > 0 \}$

$= \{ e^{i(3\pi/4)}, 2e^{i(3\pi/4)}, 3e^{i(3\pi/4)}, \dots \}$



$\log\left(re^{i(\frac{3\pi}{4})}\right) = \ln\left|re^{i(\frac{3\pi}{4})}\right| + i\left(\frac{3\pi}{4}\right)$

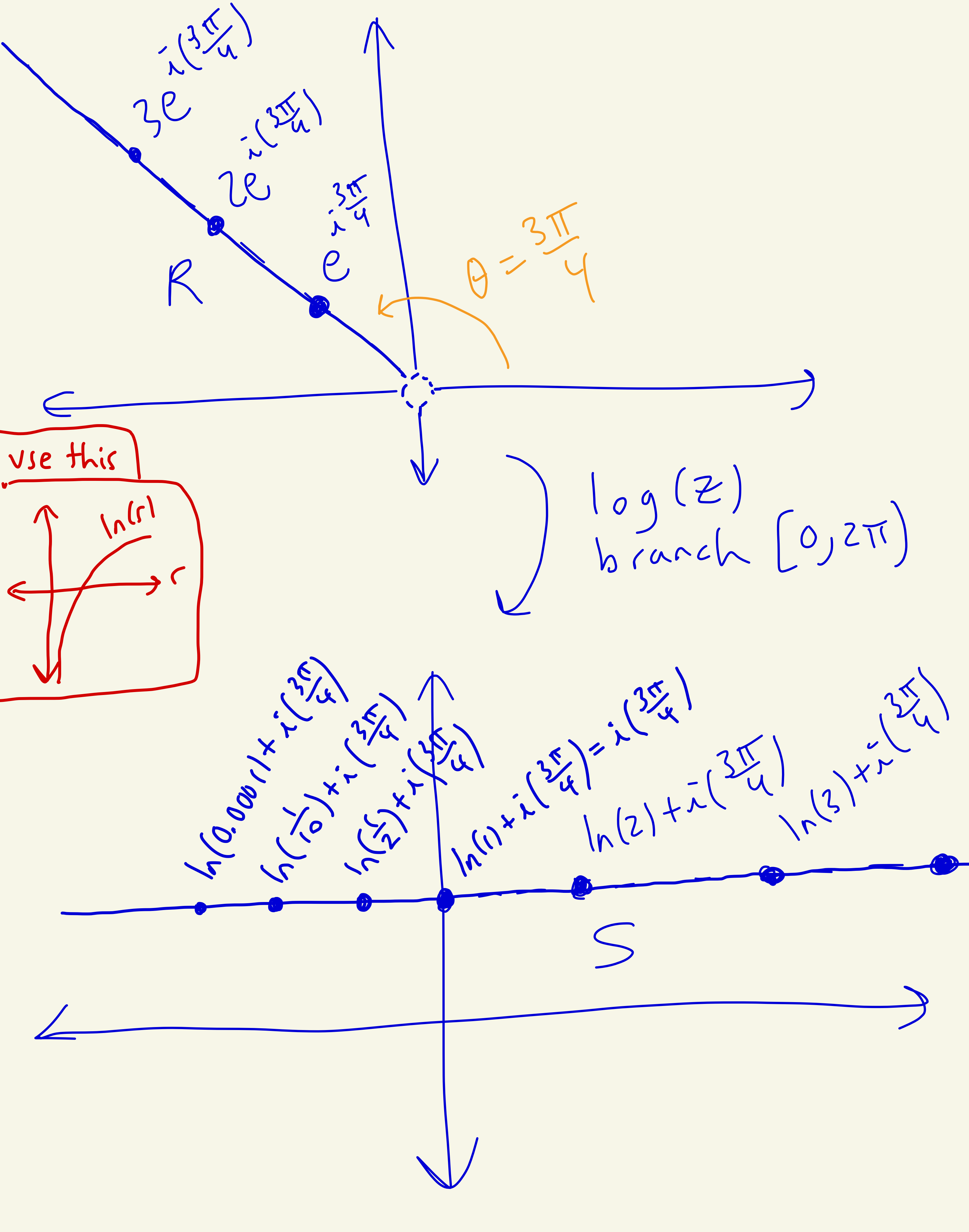
$= \ln(r) + i\left(\frac{3\pi}{4}\right)$

choose branch $[0, 2\pi)$

This branch of log sends R to

$S = \{ \ln(r) + i(\frac{3\pi}{4}) \mid r \in \mathbb{Z}, r > 0 \}$

(next page has pic)



$3e^{i\left(\frac{3\pi}{4}\right)}$
$2e^{i\left(\frac{3\pi}{4}\right)}$
$e^{i\frac{3\pi}{4}}$
R
$\theta = \frac{3\pi}{4}$
use this
ln(r)
r
log(z)
branch [0, 2π)
$\ln(0.0001) + i\left(\frac{3\pi}{4}\right)$
$\ln\left(\frac{1}{10}\right) + i\left(\frac{3\pi}{4}\right)$
$\ln\left(\frac{1}{2}\right) + i\left(\frac{3\pi}{4}\right)$
$\ln(1) + i\left(\frac{3\pi}{4}\right) = i\left(\frac{3\pi}{4}\right)$
$\ln(2) + i\left(\frac{3\pi}{4}\right)$
$\ln(3) + i\left(\frac{3\pi}{4}\right)$
S

8(a) $\sin^2(z)+\cos^2(z)$

$$= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2$$

$$= \frac{e^{2iz}-2+e^{-2iz}}{-4} + \frac{e^{2iz}+2+e^{-2iz}}{4}$$

$$= \frac{-e^{2iz}+2-e^{-2iz}+e^{2iz}+2+e^{-2iz}}{4}$$

$$= \frac{4}{4} = 1$$

8(b) $\sin(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{2i} =$

$$= \frac{e^{-iz} - e^{iz}}{2i} = -\left[\frac{e^{iz} - e^{-iz}}{2i}\right]$$

$$= -\sin(z)$$

8(c) $\cos(-z) = \dfrac{e^{i(-z)} + e^{-i(-z)}}{2}$

$$= \frac{e^{-iz} + e^{iz}}{2} = \cos(z)$$

9 (a) Pick some branch of the logarithm. Then,

$$a^{b+c} = e^{(b+c)\log(a)}$$
$$= e^{b\log(a) + c\log(a)}$$
$$= e^{b\log(a)} e^{c\log(a)}$$
$$= a^b \cdot a^c$$

in class we showed $e^{z+w} = e^z e^w$

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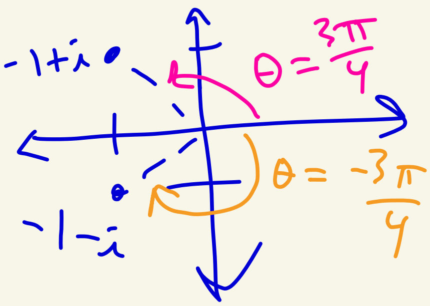
9 (b) Suppose $\log(ab) = \log(a) + \log(b)$

Then,

$$(ab)^c = e^{c\log(ab)} = e^{c\log(a) + c\log(b)}$$
$$= e^{c\log(a)} e^{c\log(b)} = a^c b^c$$

9(c) Pick the branch of log corresponding to $[-\pi, \pi)$. Then,

$$\log(i(-1+i)) = \log(-1-i)$$
$$= \ln|-1-i| + i\left(-\frac{3\pi}{4}\right)$$
$$= \ln(\sqrt{2}) - \frac{3\pi}{4} i$$



$$\log(i) = \ln|i| + i\left(\frac{\pi}{2}\right) = i\frac{\pi}{2}$$

$$\log(-1+i) = \ln|-1+i| + i\left(\frac{3\pi}{4}\right)$$
$$= \ln(\sqrt{2}) + i\left(\frac{3\pi}{4}\right).$$

So, here if $a = i$, $b = -1+i$, then

$$\log(ab) = \ln(\sqrt{2}) - \frac{3\pi}{4} i$$

$$\log(a) + \log(b) = \ln(\sqrt{2}) + i\left(\frac{5\pi}{4}\right)$$

differ by $2\pi i$

So, $\log(ab) \neq \log(a) + \log(b)$.

They differ by $2\pi i$.

In this case, $(i(-1+i))^{1/2} =$

$$= (ab)^{1/2} = e^{\frac{1}{2}\log(ab)} = e^{\frac{1}{2}\left[\ln(\sqrt{2}) - \frac{3\pi}{4} i\right]}$$

$$= e^{(1/2)\ln(\sqrt{2})}\, e^{-\frac{3\pi}{8} i}$$

$$= \boxed{e^{\ln(\sqrt{\sqrt{2}})}\, e^{i(-\frac{3\pi}{8})}}$$

$m\ln(n) = \ln(n^m)$ for real $\ln(x)$ function

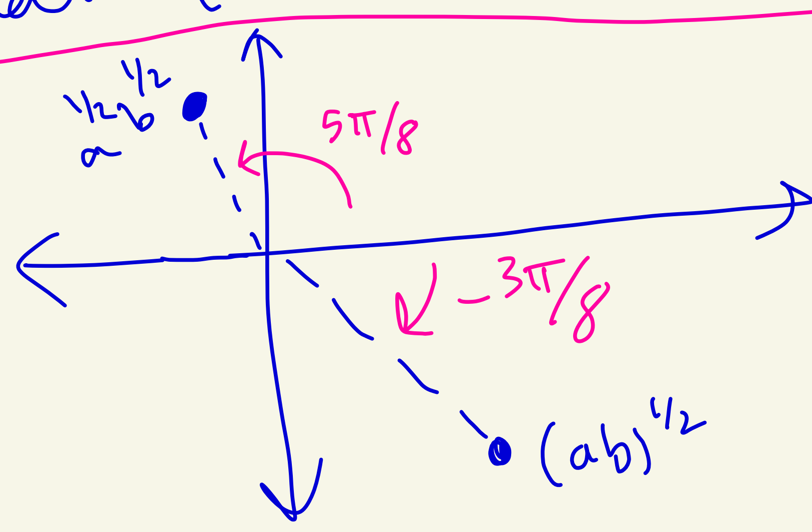
While, $(i)^{1/2}(-1+i)^{1/2} =$

$$= a^{1/2}\, b^{1/2} = e^{\frac{1}{2}\log(a)}\, e^{\frac{1}{2}\log(b)}$$

$$= e^{\frac{1}{2}(i\frac{\pi}{2})}\, e^{\frac{1}{2}(\ln\sqrt{2} + i(\frac{3\pi}{4}))} = \boxed{e^{\ln(\sqrt{\sqrt{2}})}\, e^{i(\frac{5\pi}{8})}}$$

Note that $(ab)^{1/2} \neq a^{1/2} b^{1/2}$.

see picture



10 No.

$$\left|\sin(i)\right| = \left|\frac{e^{i(i)} - e^{-i(i)}}{2i}\right|$$

$$= \left|\frac{e^{-1} - e^{1}}{2i}\right|$$

$|2i| = |2||i|$
$= |2|$

$$= \frac{\left|\frac{1}{e} - e\right|}{2} \approx 1.1752$$

(11) Let $z = x + iy$.

Then, $\sin(z) = 0$

iff $\dfrac{e^{iz} - e^{-iz}}{2i} = 0$

iff $e^{iz} - e^{-iz} = 0$

iff $e^{i(x+iy)} - e^{-i(x+iy)} = 0$

iff $e^{-y} e^{ix} - e^{y} e^{-ix} = 0$

iff $e^{-y}\left[\cos(x) + i\sin(x)\right] - e^{y}\left[\underbrace{\cos(-x)}_{\cos(x)} + \underbrace{i\sin(-x)}_{-i\sin(x)}\right] = 0$

iff $\underbrace{(e^{-y} - e^{y})\cos(x)}_{\text{real part}} + i\,\underbrace{(e^{-y} + e^{y})\sin(x)}_{\text{imaginary part}} = 0$

iff $\boxed{\begin{array}{l} \text{both } (e^{-y} - e^{y})\cos(x) = 0 \quad (*) \\ \text{and } (e^{-y} + e^{y})\sin(x) = 0 \quad (**) \end{array}}$

So we need both (*) and (**) to be true to get $\sin(z) = 0$.

Let's start with (**) because as you will see its easier.

Note that

$$(e^{-y}+e^{y})\sin(x)=0$$

iff $(e^{-y}+e^{y})=0$ or $\sin(x)=0$

But $e^{-y}+e^{y}>0$ since $e^{-y}>0$ and $e^{y}>0$.

Thus, $e^{-y}+e^{y}\neq 0$ for all y.

Thus, we need $\sin(x)=0$.

This implies $x=\pi n$ where $n\in\mathbb{Z}$, ie $n=0,\pm1,\pm2,\dots$

Now plug this into (*) to get

$$(e^{-y}-e^{y})\cos(\pi n)=0$$

But $\cos(\pi n)\neq 0$ for any $n\in\mathbb{Z}$.

So we need $e^{-y}-e^{y}=0$.

And $e^{-y}-e^{y}=0$ iff $e^{-y}=e^{y}$ iff $1=e^{2y}$

multiply by e^{y}

And $1=e^{2y}$ iff $y=0$.

Thus, $\Rightarrow$

Thus, $\sin(z)=0$

iff $y=0$ and $x=\pi n$ where $n\in\mathbb{Z}$

iff $z=x+iy=0+\pi n$ where $n\in\mathbb{Z}$

iff $z=\pi n$ where $n\in\mathbb{Z}$.

(12) Suppose $\log(z)$ corresponds to $-\pi \leq \arg(z) < \pi$.

Let $z = re^{i\theta}$ with $-\pi \leq \theta < \pi$ and $r \neq 0$.

Then,

$\log(z) = 0$ iff $\ln|z| + i\arg(z) = 0$

iff $\ln(r) + i\theta = 0$.

iff $r = 1$ and $\theta = 0$

iff $z = 1 \cdot e^{i0} = 1$. $\boxtimes$