Math 4680 - Homework # 2

Elementary Functions

1. Express the following in the form $x + iy$.
   (a) e^{2+i}
   (b) $\sin(2 - i)$
   (c) $e^{3-\pi i}$
   (d) $\cos(3\pi + i)$.

2. For each problem, first find all possible values for the expression. Then calculate the value when the branch of the argument is $[0, 2\pi)$ and when the branch of the argument is $[-\pi, \pi)$.
   (a) $\log(1)$
   (b) $\log(i)$
   (c) $\log(-1)$
   (d) $\log(1 + i)$
   (e) $(-i)^i$
   (f) $(-1)^i$
   (g) 2^i

3. Find all the values of $z \in \mathbb{C}$ that solve the following equations.
   (a) $e^z = -1$
   (b) $e^z = 1 + i\sqrt{3}$
   (c) $\cos(z) = 4$
   (d) $\sin(z) = 4$.

4. What does the function $f(z) = z^3$ map the first quadrant onto?

5. What does the function $f(z) = 1/z$ map the inside of the unit circle to?

6. What does the function $f(z) = e^z$ map the following set onto?

$$S = \{x + yi \mid x \in \mathbb{R}, -\pi/4 \leq y < 0\}$$

7. What does the logarithm function do to the following set? Pick the branch of the logarithm corresponding to the argument in the range $[0, 2\pi)$.

$$R = \left\{re^{i(3\pi/4)} \mid r \in \mathbb{Z},\ r > 0\right\}$$

8. Prove the following formulas for $z \in \mathbb{C}$.

   (a) $\sin^2(z) + \cos^2(z) = 1$

   (b) $\sin(-z) = -\sin(z)$

   (c) $\cos(-z) = \cos(z)$

9. (a) Let $a, b, c \in \mathbb{C}$. Prove that $a^b a^c = a^{b+c}$ using any fixed branch of the logarithm.

   (b) Let $a, b, c \in \mathbb{C}$. Prove that $(ab)^c = a^c b^c$ if you chose a branch of the logarithm such that $\log(ab) = \log(a) + \log(b)$ (with no extra $2\pi n i$).

   (c) Find $a, b, c \in \mathbb{C}$ where $(ab)^c \neq a^c b^c$. [Hint: Start by choosing a branch of the logarithm so that $\log(ab) \neq \log(a) + \log(b)$ for your chosen a, b, c.]

10. Is it true that $|\sin(z)| \leq 1$ for all $z \in \mathbb{C}$?

11. Prove that $\sin(z) = 0$ if and only if $z = n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$.

12. Let $z \in \mathbb{C}$ with $z \neq 0$. Prove that $\log(z) = 0$ if and only if $z = 1$, using the branch of the argument with $-\pi \leq \arg(z) < \pi$.