$\boxed{7.3}$

(1) Suppose $\varphi : 2\mathbb{Z} \longrightarrow 3\mathbb{Z}$ is a ring homomorphism. Suppose that $\varphi(2) = 3k$. for some $k \in \mathbb{Z}$. Then,

$$\varphi(4) = \varphi(2) + \varphi(2) = 3k + 3k = 6k$$

and

$$\varphi(4) = \varphi(2)\varphi(2) = (3k)(3k) = 9k.$$

Thus, $6k = 9k$. So, $k = 0$.

Thus, $\varphi(0) = 0$ and $\varphi(2) = 0$.

So, φ is not 1-1. Thus, $2\mathbb{Z} \not\cong 3\mathbb{Z}$

as rings.

(5) Suppose $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is a ring homomorphism. Let $\varphi(1,0) = m$ and $\varphi(0,1) = n$. Given $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ we have

$$
\begin{aligned}
\varphi((a,b)) &= \varphi((a,0) + (0,b)) \\
&= \varphi((a,0)) + \varphi((0,b)) \\
&= a\,\varphi((1,0)) + b\,\varphi((0,1)) \\
&= am + bn.
\end{aligned}
$$

[You can check that any map $\varphi(a,b) = am + bn$ where $m, n \in \mathbb{Z}$ is a ring homomorphism.]

Let $\varphi(a,b) = am + bn$.

$\ker(\varphi) = \{ (a,b) \mid am + bn = 0 \}$

$\text{im}(\varphi) = \{ am + bn \mid m, n \in \mathbb{Z} \} = \{ kd \mid k \in \mathbb{Z} \}$

Where $d = \gcd(a,m)$.

⑥

(a) $\varphi\begin{pmatrix}1&1\\1&1\end{pmatrix}\varphi\begin{pmatrix}1&1\\1&1\end{pmatrix} = 1\cdot 1 = 1$

$\varphi\left(\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}\right) = \varphi\begin{pmatrix}2&2\\2&2\end{pmatrix} = 2$

So, φ is not a ring hom.

(b) $\varphi\begin{pmatrix}2&0\\0&2\end{pmatrix}\varphi\begin{pmatrix}2&0\\0&2\end{pmatrix} = 4\cdot 4 = 16$

$\varphi\left(\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}\right) = \varphi\begin{pmatrix}4&0\\0&4\end{pmatrix} = 8$

So, φ is not a ring hom.

(c) $\varphi\left(\begin{pmatrix}1&2\\2&1\end{pmatrix}\right)+\varphi\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = (1-4)+(0-1) = -4$

$\varphi\left(\begin{pmatrix}1&2\\2&1\end{pmatrix}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \varphi\begin{pmatrix}1&3\\3&1\end{pmatrix} = 1-9 = -8$

So, φ is not a ring hom.

(8) (a) Not an ideal.

$(1,1) \in \{(a,a) \mid a \in \mathbb{Z}\}$

$(1,0) \in \mathbb{Z} \times \mathbb{Z}$

but

$(1,0)(1,1) = (1,0) \notin \{(a,a) \mid a \in \mathbb{Z}\}$.

(c) Let $I = \{(2a,0) \mid a \in \mathbb{Z}\}$.

<u>I is a subgroup of $\mathbb{Z} \times \mathbb{Z}$</u>: $(0,0) \in I$.

Let $(2a,0)$ and $(2b,0) \in I$. Then,

$(2a,0) - (2b,0) = (2(a-b), 0) \in I$.

So, by the subgroup criteria, I is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

<u>I is closed under $\mathbb{Z} \times \mathbb{Z}$ multiplication</u>:

Let $(2a,0) \in I$ and $(x,y) \in \mathbb{Z} \times \mathbb{Z}$. Then,

$(x,y)(2a,0) = (2ax,0) \in I$ and

$(2a,0)(x,y) = (2ax,0) \in I$.

(17) (a) Let $s = \varphi(1_R)$. Suppose $s \neq 0_S$ (see note below)

Then, $s^2 = \varphi(1_R)\varphi(1_R) = \varphi(1_R \cdot 1_R) = \varphi(1_R) = s$.

So, $s^2 - s = 0$. So, $s(s - 1_S) = 0$.

If $s \neq 1_S$, then this says that $s = \varphi(1_R)$ is a zero divisor.

[Note: If $\varphi(1_R) = 0_S$, then $\varphi(x) = 0_S$ for all $x \in R$.
pf: $\varphi(x) = \varphi(x \cdot 1_R) = \varphi(x)\varphi(1_R) = \varphi(x)\, 0_S = 0_S$.]

So, if S is an integral domain, then $\varphi(1_R)$ cannot be a zero divisor, so $\varphi(1_R) = 1_S$.

(b) Let u be a unit of R.

Then,

$$\varphi(u)\varphi(u^{-1}) = \varphi(uu^{-1}) = \varphi(1_R) = 1_S$$

and

$$\varphi(u^{-1})\varphi(u) = \varphi(u^{-1}u) = \varphi(1_R) = 1_S.$$

So, $\varphi(u)^{-1} = \varphi(u^{-1})$. Thus, $\varphi(u)$ is a unit.

(18) (a) We proved last quarter that $I \cap J$ is a subgroup of R.

Let $r \in R$ and $x \in I \cap J$.

Then $rx \in I$ since $x \in I$ and I is an ideal.
And $rx \in J$ since $x \in J$ and J is an ideal.
Also, $xr \in I$ since $x \in I$ and I is ~~and~~ an ideal.
And $xr \in J$ since $x \in I$ and J is an ideal.

So, $rx \in I \cap J$ and $xr \in I \cap J$

24 (a) $0_R \in \varphi^{-1}(J)$ since $\varphi(0_R) \in J$. Let $x \in \varphi^{-1}(J)$ and $y \in \varphi^{-1}(J)$. So, $\varphi(x) = a$ and $\varphi(y) \in b$ where $a, b \in J$. So,

$$\varphi(x-y) = \varphi(x) - \varphi(y) = a - b \in J$$

since J is an ideal. So, $x-y \in \varphi^{-1}(J)$.

So, $\varphi^{-1}(J)$ is a subgroup of S.

Let $r \in R$ and $x \in \varphi^{-1}(J)$. So,

$\varphi(rx) = \varphi(r)\varphi(x) \in J$ since $\varphi(x) \in J$ and J is an ideal. Also,

$\varphi(xr) = \varphi(x)\varphi(r) \in J$ since $\varphi(x) \in J$ and J is an ideal.

Therefore, $\varphi^{-1}(J)$ is an ideal of R.

(24) (b) $0_R \in I$ since I is an ideal of R. So, $0_S = \varphi(0_R) \in \varphi(I)$.

Let $a, b \in \varphi(I)$. Then, $a = \varphi(x)$ and $b = \varphi(y)$ where $x, y \in I$.

So, $x - y \in I$, since I is a subgroup, and $a - b = \varphi(x-y) \in \varphi(I)$.

So, $\varphi(I)$ is a subgroup by the subgroup criteria.

Let $x \in \varphi(I)$ and $s \in S$. Since φ is onto there is an element $r \in R$ with $\varphi(r) = s$. Since $x \in \varphi(I)$, there is an element $y \in I$ with $\varphi(y) = x$. Since I is an ideal $ry \in I$ and $yr \in I$. So,

$$sx = \varphi(r)\varphi(y) = \varphi(ry) \in \varphi(I)$$

and

$$xs = \varphi(y)\varphi(r) = \varphi(yr) \in \varphi(I).$$

So, $\varphi(I)$ is an ideal of S.