Trigonometric Graphs

Graphs of Sine and Cosine

In Figure 13, we showed the graphs of $f(x) = \sin x$ and $f(x) = \cos x$, for angles from 0 rad to $2\pi$ rad. In reality these graphs extend indefinitely to the left of 0 radians (negative angles) and to the right of $2\pi$ radians, and the wavy pattern displayed by both functions repeats over and over in both directions. This means that when the domain of sine and cosine are all the real numbers and we write it as $(-\infty, \infty)$. In what follows, we illustrate how one can generate the graphs of these functions with the help of a unit circle.

As we saw earlier, the $x$-coordinate of a point $(x,y)$ on a unit circle is the horizontal distance from $y$–axis to the point, and that the $y$-coordinate is the vertical distance from the $x$-axis to the point.

In all figures similar to Figure 13 that will be shown in this section, a blue line on the unit circle will represent the horizontal distance from $y$-axis to the point $(x,y)$ on the unit circle. Similarly, a red line will represent the vertical distance from the $x$-axis to the point $(x,y)$.

Moreover, since the $x$-coordinate and the $y$-coordinate of a point on the unit circle are equal to the values of cosine and sine of the associated angle, respectively, a red line and a blue line will also represent the value of $\sin x$ and $\cos x$. Green lines on the unit circle, and on the $x$-axis of both functions, will represent the angle in radians, while gray lines (not the one on the unit circle) will show the graph of these functions.

We begin with Figure 14, which shows the same layout displayed by Figure 13.
Figure 14. Point (1,0) on the unit circle is the point corresponding to angle 0 rad.

Note that point (1,0) on the unit circle of Figure 14 is shown by means of a small gray circle. This point is the corresponding point to angle 0 radians. The coordinates of point (1,0) are equal to the values of $\cos 0$ and $\sin 0$, respectively. Therefore, the blue lines displayed on this figure represent not only the distance from (1,0) to the y-axis but also the value of $\cos 0$, which is equal to 1. There is no red line shown in Figure 14 because (1,0) lies on the x-axis, and therefore the distance from (1,0) to the x-axis is zero which agrees with the fact that $\sin 0 = 0$.

Our goal is to explain how the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ shown in Figure 13 were generated using the coordinates of the points on the unit circle. We begin with Figure 15.
Figure 15. Graphs \( f(x) = \sin x \) and \( f(x) = \cos x \) from 0 rad to \( \frac{\pi}{4} \) rad.

Notice that the angles used for making these graphs (represented by the green line) belong to the interval 0 to \( \frac{\pi}{4} \) rad. As the angle gradually changes from 0 to \( \frac{\pi}{4} \) rad, the point on the unit circle moves counterclockwise from an initial position \((1,0)\) to a terminal position \(\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\). During this process the red line, which has an initial length of zero at angle 0 rad, gradually increases to a length of \(\frac{\sqrt{2}}{2}\). In a similar manner, the blue line which has an initial length of 1 at 0 rad, gradually decreases to a length of \(\frac{\sqrt{2}}{2}\). The gradual increase and decrease of these lines as we change the angle is recorded by the gray lines. These gray lines represent the graphs of \( \sin x \) and \( \cos x \) from 0 rad to \( \frac{\pi}{4} \) rad.

The process just explained for the graphs in Figure 15 can be generalized to bigger angle intervals. The following three figures show the graphs of \( f(x) = \sin x \) and \( f(x) = \cos x \) as the angle interval is increased from 0 rad to \( \frac{\pi}{2} \) rad, to 0 rad to \( \pi \) rad, and to 0 rad to \( \frac{3\pi}{2} \) rad, respectively.
Illustrating $\sin(x)$ and $\cos(x)$
The wavy pattern in both of these functions is due to the fact that the coordinates of the point on the unit circle gradually change in value from -1 to 1 as this point moves around the unit circle as we change the angle.

Furthermore, the wavy pattern of these functions repeats itself every $2\pi$ rad, which follows directly from the derivation using the unit circle. We know that a point on the unit circle moves as we change the angle. After a complete rotation the angle has changed in value by $\pm 2\pi$ rad depending on the direction chosen, but the point on the unit circle corresponding to the new angle is the same as the initial one, and so is the value of the function. More generally, regardless of where we start on the graph, the value of these functions is the same for angles separated by multiples of $\pm 2\pi$ rad.

The sign of these functions can also be explained in terms of a unit circle. Let’s consider the function $f(x) = \sin x$ whose value is given by the $y$-coordinate of the points on the unit circle. For angles between 0 and $\pi$ rad, any corresponding point on the unit circle has a positive $y$-coordinate, and thus the value of $f(x) = \sin x$ is positive. For angles between $\pi$ and $2\pi$ rad, the $y$-coordinate of any corresponding point on the unit circle is negative, and consequently the value of this function is negative. A similar analysis can be made for $f(x) = \cos x$. 
Exercises.

1. Convert the given angle in degrees to radians and revolutions.
   (a) 110°, (b) 510°, (c) −260°, (d) −660°.

2. Convert the given angle in radians to degrees and revolutions.
   (a) 10\pi\text{ rad}, (b) 4.5\pi\text{ rad}, (c) −2.5\pi\text{ rad}, (d) −10\text{ rad}.

3. For the given angle, which is measured with respect to the positive x-axis, find a positive angle and a negative angle whose terminal side coincides with the terminal side of the given angle.
   (a) 110°, (b) −1.5\pi\text{ rad}.

4. Find the angle defined by a circular arc of length 10 cm on a circle of radius 5 cm.

5. Sketch a right triangle that has acute angle \( \theta \) and the given side, and then find the six trigonometric ratios.
   (a) \( \theta = 30^\circ \), hypotenuse = 2, (b) \( \theta = 45^\circ \), adjacent side = 2.

6. Evaluate the expression without using a calculator.
   (a) \( \sin \frac{5\pi}{6} + \cos \frac{2\pi}{3} \), (b) \( \tan 330^\circ - \csc 210^\circ \).

7. Find the reference angle of the given angle.
   (a) \( \theta = 222^\circ \), (b) \( \theta = -\frac{2\pi}{3} \).

8. For the given angle \( \theta \), find the quadrant of its terminal side.
   (a) \( \theta = 7.5\text{ rev} \), (b) \( \theta = 7.5\pi\text{ rad} \), (c) 750°.

9. Determine whether or not the given point belongs to the unit circle.
   (a) \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \), (b) \( \left( \frac{1}{2}, \frac{3}{2} \right) \), (c) \( \left( \frac{1}{2}, -\frac{\sqrt{2}}{2} \right) \).

10. Find the missing coordinate of the point, using the fact that the point lies on the unit circle in the given quadrant.
    (a) \( \left( -\frac{\sqrt{3}}{2}, \right) \) and QIII, (b) \( \left( -\frac{1}{2}, \right) \) and QIV.

11. Find the corresponding point for the given angle \( \theta \).
    (a) \( \theta = \frac{5\pi}{6} \), (b) \( \theta = -135^\circ \).

12. Find an approximate value of the given trigonometric function using a calculator.
    (a) \( \sin(-233^\circ) \), (b) \( \cos \left( \frac{50\pi}{3} \right) \).