Trigonometry of Right Triangles

Right Triangles
A right triangle, as the one shown in Figure 5, is a triangle that has one angle measuring 90°. The side opposite to the right angle is the longest of the three sides and it is called the hypotenuse. Since the sum of the three angles has to be 180° (in plane geometry), the other two angles are acute angles (less than 90°). If you label one of the acute angles $\theta$, then the side of the triangle next to the angle $\theta$ (excluding the hypotenuse) is called the adjacent side to $\theta$ and the one across from $\theta$ is called the opposite side to $\theta$.

![Diagram of right triangle showing hypotenuse, opposite side, and adjacent side with angle $\theta$.]

**Figure 5.** Components of a right triangle.

The Pythagorean Theorem
This theorem describes the relationship between the lengths of the sides of a right triangle. If $C$ represents the length of the hypotenuse and $A$ and $B$ are the lengths of the other two sides, as shown in Figure 6, then the lengths of the sides are related by the formula:

$$C^2 = A^2 + B^2$$

or equivalently by $C = \sqrt{A^2 + B^2}$. 

(3)
The usefulness of the Pythagorean Theorem is that if we know the lengths of any two of the sides of a right triangle, we can compute the length of the missing side by solving $C^2 = A^2 + B^2$ for the unknown side.

**Example 4.** Using Figure 6 as reference, assume that the length of side $A$ equals 3 cm and that of $B$ equals 4 cm. Find the length of the hypotenuse.

Solution: The length of the hypotenuse is found using $C^2 = A^2 + B^2$ by solving for $C$ and then substituting the known values in the formula, as follows:

$C^2 = A^2 + B^2$

$C = \sqrt{A^2 + B^2}$

$C = \sqrt{3^2 + 4^2} = 5$ cm.

Thus, the hypotenuse has a length of 5 cm when the other sides equal 3 cm and 4 cm, respectively.

**Example 5.** Using Figure 6 as reference, find the length of $A$ if you know that the length of the hypotenuse $C$ is 6 inches and that of side $B$ is 2 inches.

Solution: We find the length of $A$ by solving (2) for $A$. 

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*Figure 6. Illustration of the Pythagorean Theorem.*
$C^2 = A^2 + B^2$
$A^2 = C^2 - B^2$
$A = \sqrt{C^2 - B^2}$
$A = \sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}$ inches.

Thus, the length of side A is $4\sqrt{2}$ inches.

**Trigonometric Ratios**

The size of the angle $\theta$ depends on the relative lengths of the sides of a right triangle. To relate the size of $\theta$ to the length of the sides of the right triangle, we define six trigonometric ratios, one for each of the six ways that the sides of the triangle can be compared two at a time. These ratios are called sine, cosine, tangent, cosecant, secant, and cotangent of $\theta$. We denote and define these ratios as follows:

$$
\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}
$$

$$
\csc \theta = \frac{\text{hypotenuse}}{\text{opposite side}} \quad \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent side}} \quad \cot \theta = \frac{\text{adjacent side}}{\text{opposite side}}
$$

From (4), notice that $\sin \theta$ and $\csc \theta$ are reciprocal to each other, and so are $\cos \theta$ and $\sec \theta$, and $\tan \theta$ and $\cot \theta$. We can express these facts in the following identities:

$$
csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}.
$$

(5)
Moreover, if we divide $\sin \theta$ by $\cos \theta$, we get that \[
\frac{\sin \theta}{\cos \theta} = \frac{\text{opposite side}}{\text{adjacent side}},
\] which is the definition of $\tan \theta$. Similarly, if we divide $\cos \theta$ by $\sin \theta$, we get the definition of $\cot \theta$. Thus, we obtain the following identities:

\[
\begin{align*}
tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta}.
\end{align*}
\]

Looking at (5) and (6) we see that all six trigonometric ratios can be expressed in terms of sine and cosine - the two ratios that are discussed in depth in these notes. Moreover, in Section 1.8 and a few other sections of your textbook you will use sine and cosine while you study the oscillatory behavior of biological systems. These two trigonometric functions are those that occur in natural systems that have regular oscillations.

For future reference, it is a good idea to group all six trigonometric ratios in terms of sine and cosine under one single frame:

\[
\begin{align*}
\sin \theta & \quad \cos \theta \\
\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \\
\sec \theta &= \frac{1}{\cos \theta} & \csc \theta &= \frac{1}{\sin \theta}.
\end{align*}
\]

The following two examples show how to evaluate trigonometric ratios using the known sides of a right triangle.

**Example 6.** (a) Using the given right triangle, evaluate the six trigonometric ratios given in (4) for the angle $\theta$. 

Solution: Using the definitions of the ratios, we obtain

\[
\sin \theta = \frac{3}{5} \quad \cos \theta = \frac{4}{5} \\
\tan \theta = \frac{3}{4} \quad \cot \theta = \frac{4}{3} \\
\sec \theta = \frac{5}{4} \quad \csc \theta = \frac{5}{3}
\]

(b) Using the given right triangle, evaluate \( \sin \omega \) and \( \cos \omega \).

Solution: Using now the angle \( \omega \) instead, we obtain

\[
\sin \omega = \frac{4}{5} \quad \cos \omega = \frac{3}{5}
\]

**Note.** The trigonometric ratios depend on which acute angle is used.

**Example 7.** Using the information from the given right triangle, evaluate the six trigonometric ratios.

Solution: We first determine the missing side using the Pythagorean Theorem:

\[
\left(\sqrt{10}\right)^2 = 2^2 + x^2 \\
10 = 4 + x^2 \\
x^2 = 6 \\
x = \sqrt{6}
\]
Now, we determine the six trigonometric ratios using their definition in (4):

\[
\begin{align*}
\sin \theta &= \frac{\sqrt{15}}{5} & \cos \theta &= \frac{\sqrt{10}}{5} \\
\tan \theta &= \frac{\sqrt{6}}{2} & \cot \theta &= \frac{\sqrt{6}}{3} \\
\sec \theta &= \frac{\sqrt{10}}{2} & \csc \theta &= \frac{\sqrt{15}}{3}
\end{align*}
\]

The final answers above were obtained by rationalizing the denominators. For example, we obtained \( \sin \theta = \frac{\sqrt{15}}{5} \) from \( \sin \theta = \frac{\sqrt{6}}{\sqrt{10}} \) by rationalizing the denominator of the second one. That is, we got rid of \( \sqrt{10} \) in the denominator \( \sin \theta = \frac{\sqrt{6}}{\sqrt{10}} \) by multiplying both numerator and denominator by \( \sqrt{10} \). This was done as follows follows:

\[
\sin \theta = \frac{\sqrt{6}}{\sqrt{10}} \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{60}}{10} = \frac{2\sqrt{15}}{10} = \frac{\sqrt{15}}{5}.
\]

Similarly, by definition (3) \( \cos \theta = \frac{2}{\sqrt{10}} \). Rationalizing the denominator of it we get that:

\[
\cos \theta = \frac{2}{\sqrt{10}} \frac{\sqrt{10}}{\sqrt{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}.
\]

Alternatively, we could have answered this question by using the expressions in terms of sine and cosine given in (7) instead of the definitions given in (4). The next example shows how to do this.

**Example 8.** Assume that \( \sin \theta = \frac{3}{5} \) and \( \cos \theta = \frac{4}{5} \). Evaluate the trigonometric ratios \( \tan \theta \), \( \cot \theta \), \( \sec \theta \), and \( \csc \theta \) using the trigonometric identities given in (7).
Solution:

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{3\sqrt{5}}{4/5} = \frac{3}{4} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{4/5}{3\sqrt{5}} = \frac{4}{3}
\]

\[
\sec \theta = \frac{1}{\cos \theta} = \frac{1}{4/5} = \frac{5}{4} \quad \csc \theta = \frac{1}{\sin \theta} = \frac{1}{3/5} = \frac{5}{3}
\]

Note that in this problem, all we needed to use were the trigonometric identities – we did not have to compute the individual lengths of the sides of the triangle to obtain the missing ratios.

**Trigonometric Ratios of Similar Right Triangles**

In your high school geometry class, you probably learned that two similar triangles are only different in size, that is, visually they look the same but one is a smaller version of the other. Obviously, the sides of two similar right triangles are different due to the net difference in the size between the triangles. However, their corresponding angles and trigonometric ratios are the same. In what follows, we use the similar right triangles in **Figure 7** to illustrate these facts.

![Figure 7](image-url)

**Figure 7.** The trigonometric ratios of similar triangles are the same.
The following computations show that the trigonometric ratios are the same for the two triangles despite the fact that their corresponding side lengths are different:

\[
\sin \omega = \frac{\sqrt{6}}{\sqrt{10}}, \cos \omega = \frac{2}{\sqrt{10}} \quad \sin \omega = \frac{2\sqrt{6}}{2\sqrt{10}} = \frac{\sqrt{6}}{\sqrt{10}}, \cos \omega = \frac{4}{2\sqrt{10}} = \frac{2}{\sqrt{10}}
\]

(For the left triangle) 

(For the right triangle)

Note that the triangle on the right side has sides that are twice as long as the one on the left. You can imagine that the right triangle was constructed from the left one by stretching each of its sides by a factor of two. Mathematically, this corresponds to multiplying each of the sides of the left triangle by 2. This process can be generalized to create any size similar right triangle from a given one - we just have to multiply each of the sides of a given triangle by the same factor. Since all sides are multiplied by the same factor, this factor cancels out when computing any of the trigonometric ratios, making it clear that the trigonometric ratios remain the same (not just in the example given above, but in general).

Important Right Triangles

There are two special right triangles that are important in the discussion of the material that follows, and they are well worth memorizing.

The 45° – 45° – 90° Right Triangle.

A right triangle with angles 45°, 45°, 90° can be generated by cutting a square, whose sides are of length 1, along one of its diagonals. The bisection creates two right triangles of equal size, as illustrated in Figure 8 (a). Since each of the corners of the square is a right angle, bisecting the square generates two 45° angles. Note that the sum of the angles inside any of the two triangles adds up to 180°. Also, since the bisecting line forms the hypotenuse of both right triangles, using the Pythagorean Theorem (3) we find that its length is \( \sqrt{2} \). Figure 8 (b) shows a similar right triangle with a hypotenuse of
length one. This means that to generate the triangle in (b) from (a) we multiplied the sides of the original right triangle by a factor of \( \frac{1}{\sqrt{2}} \). From either one of the similar triangles we obtain the important trigonometric ratios for sine and cosine of \( \theta = 45^\circ \) (or \( \frac{\pi}{4} \) rad):

\[
\sin 45^\circ = \frac{\sqrt{2}}{2}, \quad \cos 45^\circ = \frac{\sqrt{2}}{2}.
\]

(8)

**Figure 8.** Geometric determination of \( \sin 45^\circ \) and \( \cos 45^\circ \).

**The 30° – 60° – 90° Right Triangle.**

Consider an equilateral triangle (a triangle of equal sides) of side length 1. Suppose that we cut this triangle into two equal halves as illustrated in **Figure 9**. The bisection creates two triangles with angles \( 30^\circ, 60^\circ, 90^\circ \). The sides of each triangle are 1, \( \frac{1}{2} \), and \( \frac{\sqrt{3}}{2} \) (we used the Pythagorean Theorem (3) to find the last value).

Applying the definitions for sine and cosine in (4), we obtain the following important trigonometric ratios for the angles \( 30^\circ, 60^\circ \):

\[
\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}
\]

(9)
Using (7) we can compute some of the other trigonometric ratios for the special angles of 30°, 45°, and 60°.

**Example 9.** Find $\tan 30^\circ$, $\sec 60^\circ$, and $\csc 45^\circ$.

Solution: Recall from (7) that all trigonometric ratios can be written in terms of sine and cosine. Then:

\[
\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},
\]

\[
\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{1/2} = 2,
\]

\[
\csc 45^\circ = \frac{1}{\sin 45^\circ} = \frac{1}{1/\sqrt{2}} = \sqrt{2}.
\]

For a listing of values for sine and cosine at special angles, see the table of values in Section 1.8 (page 92) of Adler’s textbook.