Math 105: Trigonometry Notes

This set of notes includes basic trigonometry concepts that you will use in this class, and that are not included in the textbook by Adler. You must learn these concepts to ensure that you have the appropriate background for the successful completion of Math 105, Math 204, and Math 205. You can find a more in-depth description of these and other concepts in *PRECALCULUS 6e by Stewart, Redlin, and Watson*. This textbook is used in Math 104A, the prerequisite for Math 105.
Angles

Construction of an Angle.

In what follows, we use rays to describe angles. Visually we represent a ray by a straight arrow, as shown in Figure 1. You can think of a ray as a straight line that radiates (diverges) outwards from a point of origin. We will use the capital letter $R$ to denote a ray. If we want to differentiate two rays we will use subscripts, for example, $R_1$ and $R_2$.

\[ R \]

Figure 1. Illustration of a ray as a line with direction

An angle (typically labeled by a Greek letter such as $\theta$) is created by two rays $R_1$ and $R_2$ with a common point of origin $O$ called the vertex. Figure 2 shows two examples.

\[ R_2 \]

\[ \theta \]

\[ R_1 \]

\[ O \]

Figure 2. An angle $\theta$ and its components.

Think about the hands of a clock where $R_1$ and $R_2$ are the hour and minute hands respectively, and their point of rotation is the vertex $O$. During the normal operation of a clock both hands rotate in what is called the clockwise direction; the rotation of the hands in the opposite direction is called the counter-clockwise direction. At any time of the day the two hands describe an angle. With this model in mind, we can think of an angle $\theta$ as the amount of rotation about the vertex
$O$ that is required to move $R_1$ onto $R_2$ or vice versa, either in a clockwise or counter-clockwise direction.

We can also think of an angle $\theta$ as the amount of rotation a single ray $R$ makes about its origin as it moves from a starting position (called its initial position) to a final position (called its terminal position). At times we will refer to the initial and terminal position as the initial and terminal side of the angle, respectively. We define $\theta$ to be a positive angle when the rotation of $R$ is counter-clockwise, and negative if it is clockwise. We typically indicate the direction of rotation by a curved arrow connecting the initial position of $R$ to its final position, as shown in Figure 3.

**Figure 3.** Illustration of a positive and a negative angle $\theta$.

### Definition of a Angle and Units

An angle $\theta$ is the amount of rotation that a ray $R$ makes as it rotates about its origin from its initial position to its terminal position. The amount of rotation (angle) can be quantified (measured) using different units of angle measure. You are already familiar with the concept of units for example when it comes to distance and time. There are several units for distance like meters, yards, miles, and others. Similarly, for time you use seconds, minutes, hours, and other units. To measure angles, the most commonly used units are degrees (°), radians (rad), and
revolutions (rev). There are other units of angle measure like the gradian (grad) that are not commonly used in the U.S but are popular in some of the European countries such as France. The definitions of degree, radian, and revolution and their correspondences are discussed next.

A. Definition of a Degree

Suppose a ray $R$ makes a complete rotation about its origin. During this process, ray $R$ returns to its original position. Let's divide the angle corresponding to one complete rotation of $R$ into 360 steps. To visualize this, think of a circular pizza pie that is cut into 360 slices. The angle corresponding to one of the 360 slices corresponds to one degree, and it is denoted by $\theta = 1^\circ$. The whole pizza (a complete rotation of $R$) corresponds to $\theta = 360^\circ$, half a pizza to $\theta = 180^\circ$, and a quarter of a pizza to $\theta = 90^\circ$.

Why do we use 360 and not another number? Evidence shows that the use of the number 360 originated and became prominent in ancient times. The Egyptians were avid astronomers and amongst the first cultures to develop a 360 days solar calendar. This number is a very good approximation of the number of days (364-365 days) that it takes the Earth to make a complete rotation around the Sun. Moreover, the Babylonians used a base 60 numeration system (called sexagesimal), although some historians believe that 360 was the original base, which was later reduced to 60$^1$.

The number 360 is a very interesting number because it has a large number of proper divisors (numbers that divide 360 exactly); there are 24 and the list includes every number from 1 through 10 with the exception of 7. Consequently, dividing a circle into 360 equal parts it allows important fractions of a circle to be written as integers. For example, $1/2$ of 360 corresponds to 180, $1/4$ of 360 corresponds to 90, and $1/8$ of 360 corresponds to 45, etc.
B. Definition of a Radian
Suppose that ray \( R \) of length \( r \) is centered at the origin of a rectangular coordinate system and that it has an initial position parallel to the positive \( x \)-axis. Also, suppose that we rotate ray \( R \) counterclockwise to a new position (its terminal position) shown in Figure 4. While the rotation occurs, the tip of \( R \) forms (traces out) a circular path, which we call a circular arc and denote \( s \).

The rotation of ray \( R \) from its initial to its terminal position forms a positive angle \( \theta \). If the terminal position of \( R \) is such that the length of \( s \) equals the length of \( r \), then we say that angle \( \theta \) measures 1 radian. We will denote radians by rad. The length of the circular arc is commonly known as the arc length. Notice that here the initial position of \( R \) was chosen to be the positive \( x \)-axis but in reality it can be located anywhere relative to the coordinate system. The following example illustrates the definition of a radian: If \( r = 2 \text{ cm} \) (which equals the length of \( R \)) and assume that the tip of \( R \) traces out a circular arc of length \( s = 2 \text{ cm} \), then \( R \) describes an angle \( \theta = 1 \text{ rad} \).

In general, any angle \( \theta \) in radians can be computed from \( r \) and \( s \) using the formula:

\[
\theta = \frac{s}{r}.
\]  

Figure 4. For an angle \( \theta = 1 \text{ rad} \) the length of \( s \) equals the length of \( r \).
When the arc length \( s \) is half the length of \( r \) then \( \theta = \frac{1}{2} \) rad. If \( s \) is three times as long as the length of \( r \) then \( \theta = 3 \) rad. Moreover, if ray \( R \) circumscribes a complete circle (goes around once), the arc length \( s \) equals the circumference of the circle. The circumference of a circle of radius \( r \) is equal to \( 2\pi r \), where \( \pi = 3.1416 \). This means that the arc length \( s \) of a circle of radius \( r \) is equal to \( 2\pi r \), and therefore, using the formula (1), we get that \( \theta = 2\pi \) rad.

C. **Definition of Revolution (Cycle, or Oscillations)**

When a ray \( R \) centered at the origin of a coordinate system makes a complete rotation about its origin it generates an angle \( \theta \) equal to 1 revolution and we write \( \theta = 1 \) rev. \( R \) can rotate a fraction of a complete rotation, make several complete rotations, or anything in between. For example, if \( R \) makes a quarter of a complete rotation we say that \( \theta = \frac{1}{4} \) rev, and if \( R \) moves two complete rotations we say that \( \theta = 2 \) rev.

This unit of angle measure is commonly used when describing rotational speeds. For example, if a fan rotates with the speed of 100 rpm (revolutions per minute) this means that it makes 100 complete rotations in a time interval of one minute. In some applications, like in the harmonic motion of objects, instead of using the word revolution, the word cycle or oscillation is used instead. These three words can be used interchangeably because they represent the same thing, that is:

\[
1 \text{ revolution} = 1 \text{ cycle} = 1 \text{ oscillation}.
\]

Similarly, \( 1/2 \) rev = \( 1/2 \) cyc = \( 1/2 \) osc, \( 1/4 \) rev = \( 1/4 \) cyc = \( 1/4 \) osc, and so on.
Relationship Between Commonly Used Angles

There are commonly used angles that you must become familiar with and make an effort to learn their correspondences in different units. For example, you must learn that a complete rotation of $R$ corresponds to 1 rev, 360°, and $2\pi$ rad. Table 1 shows the most commonly used angles and their equivalence in degrees, radians, and revolutions.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Degrees</th>
<th>Radians</th>
<th>Revolutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>360°</td>
<td>$2\pi$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>270°</td>
<td>$\frac{3}{2}\pi$</td>
<td>$\frac{3}{4}$</td>
<td></td>
</tr>
<tr>
<td>180°</td>
<td>$\pi$</td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>90°</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{1}{6}$</td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{1}{8}$</td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{12}$</td>
<td></td>
</tr>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Commonly used angles given in different units of angle measure.

Basic Identities

Sometimes you will need to convert an angle between degrees, radians and revolutions. To do this you must use the basic identities in Table 2.

<table>
<thead>
<tr>
<th>Identity</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 360° = $2\pi$ rad</td>
<td></td>
</tr>
<tr>
<td>(b) 360° = 1 rev</td>
<td></td>
</tr>
<tr>
<td>(c) $2\pi$ rad = 1 rev</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Basic identities used for converting the units of angles.

To change units from one unit to another you must multiply a given angle (old units) by an appropriate conversion factor obtained from one of the basic identities in Table 2. What results is the angle in the new units. This is summarized in the following relation:
Angle Measure

\begin{equation}
\text{(angle in old units)} \left( \text{conversion factor} \right) = \text{angle in new units}
\end{equation}

As you saw in Section 1.3 of your textbook, when converting units you must do a unit check. When a conversion factor is used correctly the old unit cancels out and the new unit remains. In the next example we will see how to use one of the basic identities in Table 2 to convert units.

**Example 1.** Convert $80^\circ$ to radians.

Using (2) and the basic identity (a) in Table 2, we get:

$$80^\circ \left( \frac{2\pi \text{ rad}}{360^\circ} \right) = 0.44 \pi \text{ rad}.$$ 

Note that the conversion factor \( \left( \frac{2\pi \text{ rad}}{360^\circ} \right) \) was obtained from the basic identity $360^\circ = 2\pi \text{ rad}$.

A unit analysis shows that we have used the correct conversion factor as the degree unit cancels and the answer is given in radians, as desired.

**Example 2.** Change $\frac{5\pi \text{ rad}}{6}$ to degrees.

$$\left( \frac{5\pi \text{ rad}}{6} \right) \left( \frac{360^\circ}{2\pi \text{ rad}} \right) = \frac{5(360^\circ)}{(6)(2)} = 150^\circ.$$ 

**Example 3.** Write $380^\circ$ in revolutions.

$$\left( 380^\circ \right) \left( \frac{1 \text{ rev}}{360^\circ} \right) = 1.05 \text{ rev}.$$