Spring 2016 Topology Comprehensive Exam
Akis, Beer (chair), Fuller

Please do any FIVE of the seven problems below. They are worth 20 points each. Indicate CLEARLY
which five you want us to grade; otherwise, if you do more than five problems, we will select five to grade,
and they may not be the five that you want us to grade.

In the sequel, \( \mathbb{R} \) denotes the real numbers and \( \mathbb{N} \) denotes the positive integers. The usual topology on \( \mathbb{R} \)
will be denoted by \( \tau_u \). The closure of a subset \( A \) of a topological space \((X, \tau)\) will be denoted by \( \text{cl}(A) \).

1. Prove that \( (\mathbb{R}, \tau_u) \) is a connected topological space.

2. Let \( (X, \tau) \) and \( (Y, \sigma) \) both be compact topological spaces. Prove that \( X \times Y \) equipped with the
   product topology is compact (hint: it suffices to work with covers using basic open sets).

3. (a) What does it mean for a sequence \( \langle x_n \rangle \) in a metric space \((X, d)\) to be Cauchy?
   (b) Prove that a Cauchy sequence \( \langle x_n \rangle \) with a convergent subsequence \( \langle x_{n_k} \rangle \) must itself
       be convergent.
   (c) Show that if \( \langle x_n \rangle \) is Cauchy, then \( E := \{ x_n : n \in \mathbb{N} \} \) is a bounded subset of \( X \), that is,
       \( E \) is contained in some ball.

4. Let \( (X, \tau) \) and \( (Y, \sigma) \) be topological spaces and let \( f : X \to Y \) be continuous and onto.
   (a) Prove that if \( X \) is compact, then \( Y \) is compact.
   (b) Recall that \( D \subseteq X \) is called dense if \( \text{cl}(D) = X \). Show that if \( D \) is dense in \( X \),
       then \( f(D) \) is dense in \( Y \).

5. (a) Prove that \( \forall \alpha \in \mathbb{R} \), both \( (-\infty, \alpha) \) and \( (\alpha, \infty) \) belong the usual topology \( \tau_u \).
   (b) Suppose \((X, \tau)\) is a topological space and \( f : X \to \mathbb{R} \). Show that \( f \) is continuous
       iff \( \forall \alpha \in \mathbb{R} \), both \( f^{-1}((-\infty, \alpha)) \) and \( f^{-1}((\alpha, \infty)) \) are in \( \tau \). Here, \( \mathbb{R} \) is equipped with \( \tau_u \).

6. Let \( C \) be a nonempty connected subspace of a topological space \((X, \tau)\). Show that if \( C \subseteq D \subseteq \text{cl}(C) \),
   then \( D \) is connected as well.

7. Recall that \((X, \tau)\) is called regular if whenever \( p \in V \in \tau \), there exists \( W \in \tau \) with \( p \in W \subseteq \text{cl}(W) \subseteq V \).
   (a) Prove that \((X, \tau)\) is regular iff whenever \( A \) is a nonempty closed subset and \( p \notin A \), there
       exists disjoint \( V, W \) in \( \tau \) with \( p \in W \) and \( A \subseteq V \).
   (b) Prove using part (a) that each compact Hausdorff space is regular.