1. Let \( I \) be an infinite set. For each \( \alpha \in I \), let \( X_\alpha = \{0,1\} \) in the discrete topology. For every \( S \) subset of \( I \), define \( f_S = \langle x_\alpha \rangle_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \), where \( x_\alpha = 1 \) if \( \alpha \in S \), and \( x_\alpha = 0 \) if \( \alpha \notin S \).

Let \( A = \{ f_S : S \text{ is finite} \} \). Show that \( f_I \in \overline{A} \), where \( \overline{A} \) is the closure of \( A \) in the product space \( \prod_{\alpha \in I} X_\alpha \).

2. Let \( T_1, T_2 \) be topologies on a set \( X \). Prove that \( \{ U \cap V : U \in T_1, V \in T_2 \} \) is a basis for the coarsest topology on \( X \) containing \( T_1 \cup T_2 \).

3. Let \( X \) be a topological space. Let \( A, B, \) and \( C \) be connected subsets of \( X \) such that \( A \cap B = A \cap C = \emptyset \).

   (a) Recall that we say a set is “clopen” if it is both open and closed. Prove that if \( U \) is a proper nonempty clopen subset of \( A \cup B \), then \( U = A \) or \( U = B \).

   (b) Suppose that \( A \cup B \) is disconnected. Prove that if \( A \cup B \) is homeomorphic to \( A \cup C \), then \( B \) is homeomorphic to \( C \). Hint: This is the second part of a two-part question.

4. (a) Prove that each metric space \( (X,d) \) is first countable.

   (b) Give an example of a topological space \( (X,T) \) that is not first countable. Prove that your answer is correct.

5. (a) Let \( A, B \) be subsets of a topological space \( (X,T) \). Prove \( \overline{A \cup B} = \overline{A} \cup \overline{B} \), i.e. the closure of the union of \( A \) and \( B \) equals the union of their closures.

   (b) Suppose \( A \) and \( B \) have compact closures. Prove that \( \overline{A \cup B} \) is also compact.
6. (a) Prove the continuous image of a compact topological space is compact.

(b) Suppose \((X, T)\) is a compact and connected topological space. Show that the image of every non-constant, continuous, real valued function \(f : X \to R\), is a closed interval.

7. We call \(p\) a cluster point of a sequence \(\langle x_n \rangle\) in a metric space \((X, d)\), if for all \(\epsilon > 0\) and for every natural number \(k\), there exists \(n > k\) with \(d(x_n, p) < \epsilon\).

(a) Prove that the set of cluster points of a sequence \(\langle x_n \rangle\) in a metric space \((X, d)\), is closed.

(b) Give an example of a sequence of real numbers whose cluster point set is the closed interval \([0,1]\).