1. (a) Let $X$ be a set. Prove that the intersection of any nonempty collection of topologies on $X$ is a topology.

(b) Give an example to show that the union of two topologies need not be a topology.

2. Let $X$ be a normal Hausdorff space. Let $R$ denote the set of real numbers, endowed with the usual topology. We say that a subset $U$ of $X$ is a cozero set if there exists a continuous function $f : X \to R$ such that

$U = \{ x \in X : f(x) \neq 0 \}$. Prove that the collection of all cozero sets forms a basis for the given topology of $X$.

3. (a) Let $\{K_1, K_2, \ldots, K_n\}$ be a family of nonempty compact subsets of a topological space $X$. Prove $\bigcup_{j=1}^{n} K_j$ is compact.

(b) Let $F$ be a nonempty closed subset of a compact set $K \subseteq X$. Prove that $F$ is compact.

4. Consider the Euclidean plane $R^2$ equipped with the usual topology. Prove, giving complete justification of all claims, that

(a) $\{(x, y) : x^2 + y^2 > 1\}$ is a connected subset of $R^2$;

(b) $\{(x, y) : x^2 + y^2 = 1\}$ fails to be a connected subset of $R^2$. 
5. Let $\langle x_n \rangle$ be a Cauchy sequence in a metric space $(X,d)$.
   (a) Prove that $E = \{x_n : n \in \mathbb{N}\}$ is a bounded set, i.e., that $E$ is contained in a ball.
   (b) Suppose $\langle w_n \rangle$ is another sequence in the space where $\lim_{n \to \infty} d(x_n, w_n) = 0$.
   Prove that $\langle w_n \rangle$ is Cauchy as well.

6. The graph of a function $f : X \to Y$ is the set
   \[ \Gamma(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y. \]

   Suppose $X, Y$ are topological spaces and $f : X \to Y$ is a continuous function.
   Prove that $\Gamma(f)$ is path connected, if $X$ is path connected.

   Recall that a topological space $S$ is path connected, if for every $x, y \in S$ there exists a continuous function $p : [0,1] \to S$ of the unit interval into $S$, such that, $p(0) = x, p(1) = y$.

7. (a) Suppose $X, Y$ are topological spaces. Define what it means to say that a function $f : X \to Y$ is continuous.
   (b) Use your definition above to show that, for every sequence $\langle x_n \rangle \subseteq X$
       converging to $x \in X$, the sequence $\langle f(x_n) \rangle$ converges to $f(x)$.
   (c) By using your answers above, determine if the following real valued function of the real numbers (equipped with the usual topology), is continuous:

       \[ f(x) = \begin{cases} 
       0 & \text{if } x = 0, \\
       \sin(1/x) & \text{if } x \neq 0.
       \end{cases} \]