Let $\mathbb{R}$ denote the set of real numbers. Unless otherwise stated, assume that $\mathbb{R}^n$ is endowed with the usual topology, and that subsets of a given topological space are endowed with the subspace topology.

(1) (a) Define what it means for two topological spaces to be homeomorphic.

For parts (b), (c), and (d), a topological space will be given. First determine if it is homeomorphic to the line $\mathbb{R}$, and then either produce a homeomorphism (but don’t prove that it’s a homeomorphism) between that space and $\mathbb{R}$, or else explain why no such homeomorphism exists.

(b) The open interval $(0, 1) = \{t \in \mathbb{R} : 0 < t < 1\}$

(c) The plane $\mathbb{R}^2$

(d) The circle $S = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$

(2) A subset $A$ of the real line is declared open in the Sorgenfrey topology $\tau$ if $\forall a \in A$, there exists $\varepsilon > 0$ such that $[a, a + \varepsilon) \subseteq A$.

(a) Prove that the Sorgenfrey topology $\mu$ is finer than the usual topology $\tau$.

(b) Give an example of a subset $E$ which belongs to $\mu$ but does not belong to $\tau$.

(c) Show that $[0, 1]$ with the relative Sorgenfrey topology fails to be compact by producing an open cover with no finite subcover.

(3) Prove a metric space is separable if and only if it is second countable.

(4) (a) Prove that each decreasing sequence of nonempty closed sets $\langle A_n \rangle$ in a compact space $\langle X, \tau \rangle$ has nonempty intersection. (Recall that the sequence $\langle A_n \rangle$ is decreasing if $A_{j+1} \subseteq A_j$ for all $j$.)

(b) Produce a decreasing sequence of closed subsets of the Euclidean plane, each with nonempty interior, that has empty intersection.

(5) Suppose $X$ is a topological space such that every continuous function $f : X \to X$ has a fixed point. Prove that $X$ is connected. (Recall that a fixed point of a function $f$ is a point $x$ such that $f(x) = x$.)

(6) Suppose $A, B$ are subsets of a topological space $\langle X, \tau \rangle$. Prove or give a counterexample to each statement below. Here we are using the notations $\text{cl}(S)$ to denote the closure of a set $S$ and $\text{int}(S)$ to denote the interior of a set $S$.

(a) $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$

(b) $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$

(c) If $A$ is open, then $\text{int}(\text{cl}(A)) = A$.

(7) We say that a topological space $X$ has dimension zero if there exists a basis for the topology on $X$ where every basic open set is also closed in $X$. Let $\mathbb{Q}$ be the set of rational numbers. Endow $\mathbb{Q}$ with the subspace topology, as a subset of the set of real numbers with the usual topology. Prove that $\mathbb{Q}$ has dimension zero.