Instructions:

- Do exactly two problems from Part A AND two problems from Part B. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No calculators.
- Closed books and closed notes.

PART A: Do only TWO problems

1. Let

   \[ A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}. \]

   Note that \( A \) has eigenvalues \( \lambda_1 = 2 \) and \( \lambda_2 = -1 \).

   (a) [3 points] Find the eigenvectors \( v_1, v_2 \) of \( A \).

   (b) [8 points] **Without** doing any iteration, verify that the Power Method will converge to the eigenvector of \( A \) associated with \( \lambda_1 \) when we take the initial vector \( q_0 = (1, 0)^T \). Give an example of an initial vector for which it fails to converge. Briefly explain your reasoning.

   (c) [3 points] Taking the initial iterate \( q_0 = (1, 0)^T \), perform two iterations of the Power Method to obtain \( q_1, q_2 \).

   (d) [4 points] Show that if \( \langle \lambda, v \rangle \) is an eigenpair for an \( n \times n \) nonsingular matrix \( B \), then \( \langle 1/\lambda, v \rangle \) is an eigenpair for \( B^{-1} \).

   (e) [4 points] Using part (d), explain how one can modify the Power Method so that it converges to the eigenvector associated with \( \lambda_2 \) of the matrix \( A \) given above.

   (f) [3 points] Give one advantage and one disadvantage of the Power Method when used to find an approximation to the eigenvector.
2. (a) [4 points] What are the criteria for a Cholesky decomposition of a matrix $A$? 
(b) [4 points] Provide the algorithm for Cholesky decomposition. 
(c) [2 points] If the algorithm fails at any time, what can you conclude about matrix $A$? 
(d) [5 points] By going through a brief description of work involved, count how many 
flops the Cholesky algorithm requires. 
(e) [10 points] Solve the system $Ax = b$ using the Gaussian Elimination method with 
partial pivoting, where 

$$
A = \begin{pmatrix}
2 & 1 & 1 \\
2 & 2 & -1 \\
4 & -1 & 6 \\
\end{pmatrix}, \quad b = \begin{pmatrix}
3 \\
10 \\
11 \\
\end{pmatrix}.
$$

3. (a) [8 points] For 

$$
A = \begin{pmatrix}
1 & k & k \\
k & 1 & 0 \\
k & 0 & 1 \\
\end{pmatrix},
$$

where $k > 0$, show that the spectral radius of Jacobi iteration matrix is $k\sqrt{2}$. 
(b) [5 points] For what values of $k$ above does Jacobi iteration converge? What is the 
rate of convergence? 
(c) [8 points] For a given linear system $Ax = b$, and a splitting $A = M - N$, show 
that the iterative method $Mx^{(k+1)} = Nx^{(k)} + b$ converges linearly. What is the 
rate of convergence? 
(d) [4 points] If an iterative method solves a linear system $Ax = b$ with accuracy $10^{-2}$ 
in 230 iterations, then how many iterations will it need to increase accuracy to 
$10^{-3}$? Explain briefly.
PART B: Do only **TWO** problems

1. Consider the following boundary value problem (BVP)

\[ u_{xx} + u_{yy} = 4, \quad 0 < x < 1, \quad 0 < y < 1 \]

\[ u(x, 0) = x^2, \quad u(x, 1) = (x - 1)^2, \quad 0 \leq x \leq 1 \]

\[ u(0, y) = y^2, \quad u(1, y) = (y - 1)^2, \quad 0 \leq y \leq 1 \]

(a) [3 points] Show that \( u(x, y) = (x - y)^2 \) is an exact solution to the above BVP.

(b) [6 points] Show that the solution \( u(x, y) = (x - y)^2 \) is unique.

(c) [10 points] Suppose we partition the domain \([0, 1] \times [0, 1]\) into a mesh with mesh size \( h = \Delta x = \Delta y = 1/3 \). Using central differences to approximate the derivatives, write a finite difference scheme to approximate the solution \( u \) at the resulting four interior mesh points. Simplify your answer in the matrix form \( Au = b \).

(d) [6 points] Suppose the boundary condition \( u(x, 1) = (x - 1)^2 \) is changed to \( u_y(x, 1) = 2x \). Write a second order finite difference formula to approximate the solution at the point \((2/3, 1)\) in the mesh of part (c).

2. Consider the initial boundary value problem (IBVP):

\[ U_t = \alpha U_{xx} - \beta U, \quad 0 < x < 1, \quad t > 0 \]

\[ U(x, 0) = x(1 - x), \quad 0 \leq x \leq 1 \]

\[ U(0, t) = 0, \quad U(1, t) = 0, \quad t > 0 \]

where \( \alpha > 0, \beta > 0 \). An explicit approximation to the PDE with \( r = k/h^2 \) has the matrix form \( u_{j+1} = Au_j \), where \( u_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})^T \) and \( A \) is the tridiagonal matrix of order \( N - 1 \):

\[
A = \begin{pmatrix}
1 - 2r\alpha - k\beta & r\alpha & 0 & \cdots & 0 \\
-2r\alpha & 1 - 2r\alpha - k\beta & r\alpha & 0 & \vdots \\
0 & -2r\alpha & 1 - 2r\alpha - k\beta & r\alpha & 0 \\
\vdots & \ddots & \ddots & \ddots & r\alpha \\
0 & \cdots & 0 & r\alpha & 1 - 2r\alpha - k\beta
\end{pmatrix}
\]

(a) [12 points] By determining the eigenvalues of the matrix \( A \), obtain a restriction on \( r \) that guarantees that the scheme is stable.

(b) [3 points] Write an expression for the given scheme solved for \( u_{i,j+1} \) in terms of \( u_{i-1,j}, u_{i,j}, u_{i+1,j} \).

(c) [4 points] Give a consistent approximation to each of the initial and boundary conditions. (You need **not** show that your approximations are consistent.)
(d) [3 points] The truncation error for the given scheme is $T(h) = O(h^2)$. Use this fact to explain why the given scheme is consistent with the given PDE.

(e) [3 points] Assume that the given IBVP is well-posed. Explain why you can conclude from your work in the previous parts of this problem that the given scheme converges if $r$ is restricted as found in part (a).

3. Consider the PDE

$$U_{xx} + xU_{xy} - 2x^2U_{yy} = 0$$

with initial data given on $y = 0$.

(a) [4 points] Determine all values of $x$ for which the given PDE is hyperbolic.

(b) [5 points] Determine the two characteristic directions (slopes), $dy/dx$, for the given PDE at a general point $(x, y)$.

(c) [6 points] Using the result of part (b), find the exact values of the coordinates of the point of intersection, $R$, of the characteristic curves through the points $P(1, 0)$ and $Q(2, 0)$.

(d) [2 points] Give the interval of dependence for $U(x, y)$ at the point $R$ of part (c).

(e) [4 points] Derive a consistent finite difference approximation for the $xU_{xy}$ term. (You need not show that your approximation is consistent.)

(f) [4 points] Suppose we approximate the given PDE by a consistent explicit difference scheme with $h = k = 1$. Referring to the CFL condition, explain why or why not this scheme converges at the point $R$ of part (c).