PART A (Do two problems)

A-1 Consider the following elliptic boundary-value problem in the region $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$:

$$ u_{xx} + u_{yy} = 0 \quad 0 < x < 1, 0 < y < 1 $$

$$ u(0, y) = -y^2, \quad u(1, y) = 1 - y^2, \quad 0 \leq y \leq 1 $$

$$ u(x, 0) = x^2, \quad u(x, 1) = x^2 - 1, \quad 0 \leq x \leq 1 $$

a. Show that $u(x, y) = x^2 - y^2$ is an exact solution of this boundary-value problem. [5%]

b. What are the maximum and minimum values achieved by the solution, $u$, to the given boundary-value problem in the region $D$? At what points $(x, y)$ do they occur? [4%]

c. With $\Delta x = \Delta y = 1/3$, use the usual five-point difference scheme for approximating the given PDE to obtain a system of linear equations for solving this problem. Express this system in the form $Au = b$, where $A$ is a $4 \times 4$ matrix. [12%]

d. Explain why the solution to your difference approximation in part c is unique. [4%]
A-2 Consider the following difference approximation to

\[ u_t = c u_{xx} \quad (\text{where } c > 0) \quad 0 < x < 1, \ t > 0 \]

\[ u(x, 0) = x(1 - x) \quad 0 < x < 1 \]

\[ u(0, t) = 0, \ u(1, t) = 0 \quad t > 0 \]

\[ \frac{u_{i,j+1} - u_{i,j}}{k} = c \left[ \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \right], \quad \text{where } u_{i,j} = u(ih, jk) \]

a. Is this an explicit or implicit scheme? [2%]

b. If this scheme is written as \( B u_{j+1} = C u_j \), where

\[ u_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j}) \]

taking \( r = k/h^2 \), determine the matrices \( B \) and \( C \). [8%]

c. Use the Neumann (Fourier) method to determine all values of \( r = k/h^2 \) for which this scheme is stable. [12%]

d. Assume that this scheme is consistent with the given PDE. Is the scheme convergent? Why or why not? [3%]

A-3 Given the hyperbolic initial value problem

\[ \begin{cases} u_{tt} - 9u_{xx} = 0, & (-\infty \leq x \leq \infty, \ t > 0) \\ u(x, 0) = f(x), \ u_t(x, 0) = g(x), & (-\infty \leq x \leq \infty) \end{cases} \]

where \( f(x) \) and \( g(x) \) are given continuous functions.

a. Derive an explicit finite difference scheme with \( u_{i,j} = u(ih, jk) \) (\( h = \Delta x, \ k = \Delta t \), and taking \( r = k/h \)), solved for \( u_{i,j+1} \), for obtaining approximate solutions to this problem. Explain how to use this scheme to compute values along the “first row”; that is, when \( t = k \). [12%]

b. Find the characteristic curves of the given PDE through the point \( (0, \frac{1}{2}) \). [6%]

c. State the Courant-Friedrichs-Levy (C.F.L) condition. What values of \( r = k/h \) will ensure that the C.F.L condition will be satisfied for your scheme? If \( r \) is less than this value, what conclusion can you draw concerning your scheme? [7%]
PART B  (Do two problems)

B-1  Let \( A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \), where \( a, b, c \) are real numbers with \( a > 0, c > 0 \).

a. Find the spectral radius of the Jacobi iteration matrix for \( A \). [8%]
b. Using the results of part a, give conditions on \( a, b, \) and \( c \) that ensure that Jacobi iteration will converge for the linear system \( Ax = b \) (\( b \) arbitrary). [3%]
c. Give necessary and sufficient conditions on \( a, b, \) and \( c \) that ensure that the matrix \( A \) is diagonally dominant. [3%]
d. Show that \( A \) is positive definite if and only if \( ac - b^2 > 0 \). [5%]
e. Is each statement true or false for this matrix \( A \)? [3% each]
   i. If \( A \) is diagonally dominant, then it is positive definite.
   ii. If \( A \) is positive definite, then it is diagonally dominant.

B-2  a. Let \( A = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 7 & 7 \\ -2 & -7 & 5 \end{bmatrix} \).

Find the LU decomposition of \( A \), \( A = LU \), where \( L \) is unit lower-triangular and \( U \) is upper-triangular. [8%]
b. Use your result from part a to find the LDU factorization of \( A \), where \( L \) is unit lower-triangular, \( D \) is diagonal, and \( U \) is unit upper-triangular. [3%]
c. Let \( B \) be an \( n \times n \) matrix and suppose we have obtained the LU factorization of \( B \). Determine the number of multiplications / divisions it takes to solve \( Ux = c \) by backward substitution, where \( c \) is an arbitrary \( n \)-vector. [6%]
d. Let \( B \) be an \( n \times n \) matrix. Show that if \( B \) can be factored as \( B = LU \), where \( L \) is unit lower-triangular and \( U \) is upper-triangular, then this factorization is unique. [8%]
The matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues 2 and 4 and corresponding eigenvectors $[s \ -s]^T$ and $[s \ s]^T$, respectively, where $s \neq 0$.

**a.** Apply two iterations of the Power Method to the matrix $A$ with initial vector $x^{(0)} = [1, 0]^T$ to obtain $x^{(2)}$, an approximation to the eigenvector of $A$ corresponding to eigenvalue 4.  

**b.** Will the Power Method converge in this case? Explain why or why not.

**c.** Give an example of an initial vector for which the Power Method will not converge.

**d.** Obtain the QR factorization of the matrix $A$.

**e.** Obtain the first iterate in the QR method for $A$. 