California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Linear Analysis Fall 2014
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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.

Please
(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that
your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where
directed and follow any instructions concerning identification code num-
bers.

MISCELLANEOUS FACTS

\[
\begin{align*}
\sin(a + b) &= \sin a \cos b + \cos a \sin b \\
2 \sin a \sin b &= \cos(a - b) - \cos(a + b) \\
2 \sin a \cos b &= \sin(a + b) + \sin(a - b) \\
2 \cos a \sin b &= \sin(a + b) - \sin(a - b) \\
\int \sin^2(ax) \, dx &= \frac{x}{2} - \frac{1}{4a} \sin(2ax) \\
\int \cos^2(ax) \, dx &= \frac{x}{2} + \frac{1}{4a} \sin(2ax) \\
\int e^{ax} \sin bx \, dx &= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} \\
\int e^{ax} \cos bx \, dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}
\end{align*}
\]
Fall 2014 # 1. Let \( f : \mathbb{R} \to \mathbb{R} \) be the \( 2\pi \) periodic function given on \( 0 \leq x \leq 2\pi \) by \( f(x) = 1 \) for \( 0 \leq x < \pi \) and \( f(x) = 0 \) for \( \pi \leq x < 2\pi \).

a. (8pts.) Find the Fourier series for \( f \). (Either exponential form or trigonometric form, your choice.)

b. (4pts.) Give a statement of any form of the Parseval Identity theorem.

c. (8pts.) Show that 
\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.
\]

Fall 2014 # 2. Suppose \( \mathcal{E} = \{e_1, e_2, e_3, e_4\} \) is an orthonormal basis for a Hilbert space \( \mathcal{H} \) and that \( T : \mathcal{H} \to \mathcal{H} \) is a linear operator with
\[
Te_1 = e_2; \quad Te_2 = 2e_3; \quad Te_3 = 3e_4; \quad \text{and} \quad Te_4 = 0.
\]

a. Find the matrix for \( T \) with respect to the orthonormal basis \( \mathcal{E} \).

b. Show that \( T \) is bounded as a linear operator on \( \mathcal{H} \). (Do not just quote the theorem that every linear operator on a finite dimensional normed space is bounded.)

c. Find the operator norm of \( T \) giving reasons to justify your answer.

Fall 2014 # 3. Let \( P_1 \) be the space of all polynomials \( ax + b \) of degree no more than 1 with the inner product \( \langle f, g \rangle = \int_0^2 f(t)g(t) \, dt \). (You may assume this is an inner product.)

a. Find a basis for the vector space \( P_1 \) which is orthonormal with respect to that inner product.

b. Find constants \( a \) and \( b \) which make the quantity \( \int_0^2 \left| x^3 - ax - b \right|^2 \, dx \) as small as possible.

Fall 2014 # 4. For each of the following, determine if it is a vector space over \( \mathbb{R} \). Give reasons for your answers. (\( C(\mathbb{R}, \mathbb{R}) \) is the space of all continuous real valued functions on \( \mathbb{R} \) which you may assume is a vector space. You may also assume that \( \mathbb{R}^4 \) with the usual operations is a vector space.)

a. \( A = \{ f \in C(\mathbb{R}, \mathbb{R}) : f''' \text{ exists and } f'''(x) + 3f'(x) + 5f(x) = 0, \forall x \in \mathbb{R} \} \)

b. \( B = \{ f \in C(\mathbb{R}, \mathbb{R}) : f'' \text{ exists and } f''(x) + 3f'(x) + 5f(x) = \cos x, \forall x \in \mathbb{R} \} \)

c. \( C = \) the set of all \( 2 \times 2 \) matrices with entries in \( \mathbb{R} \) with the operations of matrix addition and scalar multiplication

d. \( D = \) the set of all \( 2 \times 2 \) matrices with entries in \( \mathbb{R} \) and determinant 0 with the operations of matrix addition and scalar multiplication
Fall 2014 # 5. Let $H$ be a Hilbert space over $\mathbb{C}$ and $S$ and $T$ be bounded linear operators on $H$. Are the following true or false. If true, prove. If false, explain what’s missing or give a counterexample, where appropriate.

a. (3pts.) $T$ is self-adjoint $\Rightarrow$ $T$ is normal.
b. (3pts.) $T$ is unitary $\Rightarrow$ $T$ is normal.
c. (3pts.) $T$ is normal $\Rightarrow$ $T$ is self-adjoint.
d. (3pts.) $T$ is normal $\Rightarrow$ $T$ is unitary.
e. (4pts.) $S$ and $T$ are self-adjoint $\Rightarrow$ the product $ST$ is self-adjoint.
f. (4pts.) $S$ and $T$ are unitary $\Rightarrow$ the product $ST$ is unitary.

Fall 2014 # 6. Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Suppose $T : H \to H$ is a bounded linear operator such that $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f$ and $g$ in $H$.

a. Show that all eigenvalues of $T$ are real.
b. Show that eigenvectors of $T$ corresponding to different eigenvalues are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$.

Fall 2014 # 7. For each continuous function $f$ on the interval $[0, 1]$, define $Tf$ on $[0, 1]$ by $(Tf)(x) = e^x + \lambda \int_0^x e^{x-t} f(t) dt$

a. Find a range of values for the parameter $\lambda$ for which the transformation $T$ is a contraction with respect to the supremum norm $\| f \|_{\infty} = \sup \{| f(t) | : t \in [0,1] \}$.
b. Find a range of values for the parameter $\lambda$ for which the transformation $T$ is a contraction with respect to the $L^2$ norm $\| f \|_2 = \left( \int_0^1 | f(t) |^2 dt \right)^{1/2}$.
c. Describe the iterative process for solving the integral equation $f(x) = e^x + \lambda \int_0^x e^{x-t} f(t) dt$ specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all $x$ as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

End of Exam