Do five of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number $z$.
$\text{Im}(z)$ denotes the imaginary part of the complex number $z$.
$\bar{z}$ denotes the complex conjugate of the complex number $z$.
$|z|$ denotes the absolute value of the complex number $z$.
$C([a, b])$ denotes the space of all continuous functions on the interval $[a, b]$. If there is need to specify the possible values, $C([a, b], \mathbb{R})$ will denote the space of all continuous real valued functions on $[a, b]$ and $C([a, b], \mathbb{C})$ the space of all continuous complex valued functions.
$L^2([a, b])$ denotes the space of all functions on the interval $[a, b]$ such that $\int_{a}^{b} |f(x)|^2 \, dx < \infty$.

MISCELLANEOUS FACTS

\[
\begin{align*}
\sin(a + b) &= \sin a \cos b + \cos a \sin b & \cos(a + b) &= \cos a \cos b - \sin a \sin b \\
2 \sin a \sin b &= \cos(a - b) - \cos(a + b) & 2 \cos a \cos b &= \cos(a - b) + \cos(a + b) \\
2 \sin a \cos b &= \sin(a + b) + \sin(a - b) & 2 \cos a \sin b &= \sin(a + b) - \sin(a - b) \\
\int \sin^2(ax) \, dx &= \frac{x}{2} - \frac{1}{4a} \sin(2ax) & \int \cos^2(ax) \, dx &= \frac{x}{2} + \frac{1}{4a} \sin(2ax) \\
\int x \sin(ax) \, dx &= \frac{1}{a^2} \sin(ax) - \frac{x}{a} \cos(ax) & \int x \cos(ax) \, dx &= \frac{1}{a^2} \cos(ax) + \frac{x}{a} \sin(ax)
\end{align*}
\]
Fall 2002 # 1. Let \( f \) be defined on \([-\pi, \pi]\) by \( f(x) = \begin{cases} x + \pi, & \text{for } -\pi \leq x \leq 0 \\ \pi - x, & \text{for } 0 < x \leq \pi \end{cases} \).

a. Compute the Fourier series for \( f \) on \([-\pi, \pi]\).
(Trigonometric or exponential, your choice)

b. Show that \( 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \cdots = \frac{\pi^4}{96} \).

Fall 2002 # 2. a. Suppose \( \langle \cdot, \cdot \rangle \) is an inner product on a vector space \( V \) and \( \| \cdot \| \) is the associated norm. Show that if \( f \) and \( g \) are vectors in \( V \), then \( \| f + g \|^2 + \| f - g \|^2 = 2 \| f \|^2 + 2 \| g \|^2 \).

b. Show that there is no possible inner product on the space \( C([-\pi, \pi]) \) of continuous real valued functions on the interval \([-\pi, \pi]\) for which the uniform norm, \( \| f \|_\infty = \sup \{|f(t)| : t \in [-\pi, \pi]\} \), is the associated norm.
(Hint: What happens if one function is 0 when \( x \leq 0 \) and the other when \( x \geq 0 \)?)

Fall 2002 # 3. Let \( \{f_n\}_{n=1}^\infty \) be a sequence of continuous real valued functions on the interval \([a, b]\).

a. State definitions for each of the following:
   (i) \( f_n \to f \) pointwise on \([a, b]\)
   (i) \( f_n \to f \) uniformly on \([a, b]\)
   (i) \( f_n \to f \) with respect to the \( L^2 \)-norm (that is, in \( L^2 \)-mean) on \([a, b]\)

b. Show that if \( f_n \to f \) in \( L^2 \)-norm, then \( \lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx \).

c. Give an example in which \( f_n \to f \) pointwise on \([a, b]\) but \( \lim_{n \to \infty} \int_a^b f_n(x) \, dx \neq \int_a^b f(x) \, dx \).

d. Show that if \( f_n \to f \) uniformly on \([a, b]\), then \( f_n \to f \) in \( L^2 \)-norm on \([a, b]\).

Fall 2002 # 4. a. Show that the operator \( L = -\frac{d^2}{dx^2} \) acting on the space \( \mathcal{W} = \{ f : [0, 1] \to \mathbb{R} : f'' \text{ is continuous}, f(0) = 0, \text{ and } f(1) - f'(1) = 0 \} \)
is a symmetric operator with respect to the inner product \( \langle f, g \rangle = \int_0^1 f(t)g(t) \, dt \).

b. Show that if \( f \) and \( g \) are in \( \mathcal{W} \) with \( LF = \lambda f \), \( LG = \mu g \) and \( \mu \neq \lambda \), then \( f \) and \( g \) are orthogonal with respect to that inner product.

c. Show that there are infinitely many positive values of the number \( \lambda \) for which the problem \( LF = \lambda f \) with \( f(0) = 0 \) and \( f(1) - f'(1) = 0 \) has nonzero solutions \( f \). (You need not find the \( \lambda \)'s, but, if you don't, then say something about where they are on the positive real axis.)
(Hint: sketch the graphs of \( y = x \) and \( y = \tan x \) on the same set of axes.)
Fall 2002 # 5. Suppose $k(x, t)$ is a continuous real valued function on the square $[a, b] \times [a, b]$ such that $k(x, t) = k(t, x)$ for all $x$ and $t$ in $[a, b]$. For each continuous $f$ on $[a, b]$, let $Kf$ be defined by
\[
(Kf)(x) = \int_a^b k(x, t)f(t) \, dt
\]
Suppose $\{\phi_j\}_{j=1}^\infty$ is a complete orthonormal family of functions on $[a, b]$ with respect to the inner product $\langle f, g \rangle = \int_a^b f(t)g(t) \, dt$ with $K\phi_j = \mu_j \phi_j$. For continuous $g$ on $[a, b]$ and a nonzero number $\lambda$, consider the equation
\[
(A) \quad f(x) = g(x) + \lambda \int_a^b k(x, t)f(t) \, dt
\]

a. Show how to write the solution to equation (A) in terms of $g$, $\lambda$, $\{\phi_j\}_{j=1}^\infty$, and $\{\mu_j\}_{j=1}^\infty$ if $1/\lambda$ is not one of the $\mu_j$.

b. What happens if $1/\lambda$ is one of the $\mu_j$?

Fall 2002 # 6. Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

a. Use the Gram-Schmidt process to find an orthonormal basis for $W$.

b. Find the vector in $W$ closest to $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Call it $w$.

c. What does the Bessel inequality state for the vectors in (b) (with numerical values)?

Fall 2002 # 7. For each continuous function $f$ on the interval $[0, 1]$, let
\[
(Kf)(x) = \int_0^1 f(t) \sin \pi x \sin \pi t \, dt.
\]

a. Find a function $R(x, t; \lambda)$ such that the solution to the equation $f = g + \lambda Kf$ is given by
\[
f(x) = g(x) + \lambda \int_0^1 R(x, t; \lambda)g(t) \, dt.
\]

b. Find a function $f$ such that
\[
f(x) = 1 + \int_0^1 f(t) \sin \pi x \sin \pi t \, dt
\]
Fall 2002 # 8. For each continuous function $f$ on the interval $[0, 1]$, let

$$(Tf)(x) = x + \lambda \int_0^x f(t) \sin \pi t \, dt.$$ 

a. Find a range of values of the parameter $\lambda$ for which the transformation $T$ is a contraction with respect to the supremum norm. Justify your answer.

b. Find a range of values of the parameter $\lambda$ for which the transformation $T$ is a contraction with respect to the $L^2$ norm. Justify your answer.

c. Describe the iterative process for solving the integral equation

$$f(x) = x + \lambda \int_0^x f(t) \sin \pi t \, dt$$

specifying the transformation to be iterated and explaining how an why this leads to a solution. With $f_0(x) = 0$ for all $x$ as the starting function, compute the iterates $f_1(x)$ and $f_2(x)$.

End of Exam