T-COLORINGS AND CHROMATIC NUMBER OF DISTANCE GRAPHS

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Abstract. We present connections between T-colorings of graphs and regular vertex-coloring for distance graphs. Given a non-negative integral set T containing 0, a T-coloring of a simple graph assigns each vertex a non-negative integer (color) such that the difference of colors of adjacent vertices cannot fall in T. Let $\sigma_n(T)$ be the minimum span of a T-coloring of an n-vertex complete graph. It is known that the asymptotic coloring efficiency of T, $R(T) = lim_{n\to\infty} \frac{\sigma_n}{n}$, exists for any T. Given a positive integral set D, the distance graph $G(\mathcal{Z}, D)$ has as vertex set all integers $\mathcal{Z}$, and two vertices are adjacent if their difference is in D. We prove that the chromatic number of $G(\mathcal{Z}, D)$, denoted as $\chi(\mathcal{Z}, D)$, is an upper bound of $\lceil R(T) \rceil$, provided $D = T - \{0\}$. This connection is used in calculating $\chi_\beta(m,k)$, chromatic number of $G(Z,D)$ as $D = \{1,2,3,\ldots,m\} - \{k\}$, $m > k$. Early results about $\chi_\beta(m,k)$ were due to Eggleton, Erdős and Skilton [1985] who determined $\chi_\beta(m,k)$ as $k = 1$, partially settled the case $k = 2$, and obtained upper and lower bounds for other cases. We show that $\chi_\beta(m,k) = k$, if $m < 2k$; and $\chi_\beta(m,k) = \lceil \frac{m+k+1}{2} \rceil$, if $m \geq 2k$ and k is odd. Furthermore, complete solutions for $k = 2$ and 4, and partial solutions for other even numbers k are obtained. All the optimal proper coloring presented are periodic with smallest known periods.

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1. Introduction

The T-coloring of graphs arose from the channel assignment problem which models the efficiency of assigning an integral broadcast channel to each of several stations so that interference is avoided. Interference occurs when the separation of channels of two nearby stations falls within the T-set, a non-negative integral set containing 0. Hale [14] formulated the channel assignment problem using graphs by representing each station as a vertex and connecting any pair of nearby stations by an edge. A valid channel assignment (*i.e.* without interference) is called a T-coloring. Given a T-set, a *T-coloring* of a simple graph $G(V,E)$ is a function $f : V(G) \to \mathcal{Z}^+$ such that if $\{u,v\} \in E(G)$, then $|f(u) - f(v)| \notin T$.

The efficiency of a valid channel assignment or T-coloring f can be measured by the *span* which is the difference of the largest and smallest numbers in $\{f(v)|v \in V(G)\}$. Given T and G, the *T-span of G*, denoted by $sp_T(G)$, is the minimum span among all T-colorings of G. Let σ_n be the T-span of the complete graph on n vertices. Rabinowitz and Proulx [20] and Griggs and Liu [11] proved that the *asymptotic coloring efficiency of T*,

$$R(T) = \lim_{n\to\infty} \frac{\sigma_n}{n},$$

exists and is a rational number. In addition, Griggs and Liu [11] proved that the *optimum difference sequence* $\Delta\sigma = \{\sigma_{n+1} - \sigma_n\}_{n=1}^{\infty}$ is eventually periodic. Note that when $T = \{0\}$, T-coloring is the same as regular vertex-coloring, and it is easy to see in this case, $R(T) = 1$. For any other T-sets with at least one positive integer, $R(T) \geqslant 2$. The parameter $R(T)$ is also related to a number theory problem [1,13], namely, density of sequences with missing differences. For discussion about this relationship, we refer the reader to [11].

Given a set D of positive integers, called *distance set* or *D-set*, the *distance graph* $G(\mathcal{Z}, D)$ has as vertex set all integers $\mathcal{Z}$, and two vertices are adjacent if their absolute difference is in D. Introduced by Eggleton, Erdős, and Skilton [8], the study of distance graphs was motivated by the plane-coloring problem of finding the minimum number of colors to color $\Re^2$, all the points on the Euclidean plane, so that points with unit distance receive different colors. The plane-coloring problem is equivalent to determining $\chi(\Re^2, \{1\})$, the chromatic number of the distance graph with vertex set $\Re^2$ and $D = \{1\}$. Although it is known that $4 \leqslant \chi(\Re^2, \{1\}) \leqslant 7$ ([12, 19]), the exact value remains unknown. The chromatic number of distance graphs $\chi(\mathcal{Z}, D)$ for different D-sets has been studied extensively [2, 3, 6 - 10, 12, 23 - 27].

A direct connection between T-colorings and distance graphs is provided by the T-graphs which have been used as an effective tool in the study of T-colorings [11-13]. For a given T-set, let $D = T - \{0\}$, the *T-graph* denoted as G_T, is the complement of the subgraph of the distance graph $G(\mathcal{Z}, D)$ induced by the vertex set $\mathcal{Z}^+ \cup \{0\}$,

$$G_T = \overline{G}(\mathcal{Z}^+ \cup \{0\}, T - \{0\}),$$

where $\overline{G}$ denotes the complement graph of G. The *T-graph of order n*, denoted as G_T^n, is the subgraph of G_T induced by the first n vertices, $\{0, 1, 2, 3, \ldots, n-1\}$. Therefore, $G_T^n = \overline{G}(Z_n, T - \{0\})$, where Z_n denotes the vertex set $\{0, 1, 2, \ldots, n-1\}$.

In Section 2, we will prove for any given T-set, $\lceil R(T) \rceil$ is a lower bound of $\chi(\mathcal{Z}, D)$, provided $D = T - \{0\}$. Rabinowitz and Proulx [20] proved that the clique number (the largest number of vertices of a complete graph) of $G(\mathcal{Z}, D)$, denoted as $\omega(Z, D)$, is a lower bound of $\lceil R(T) \rceil$ when $T = D - \{0\}$. Therefore, for any D, letting $D = T - \{0\}$, we have:

$$\omega(\mathcal{Z}, D) \leqslant \lceil R(T) \rceil \leqslant \chi(\mathcal{Z}, D) \tag{1}$$

The attainability of the sharpness of both inequalities in (1) above for some families of T-sets will be presented.

Section 3 is devoted to the calculation of the chromatic number of distance graphs for the D-sets of the form $D = \{1, \ldots, m\} - \{k\}$, $k \leqslant m$. Earlier results for this family of D-sets were obtained by Eggleton, Erdős, and Skilton [8]. Denoting $\chi(\mathcal{Z}, D)$ when $D = \{0, 1, 2, \ldots, m\} - \{k\}$ by $\chi_\beta(m, k)$, the same authors solved the case $k = 1$, partially solved the case $k = 2$, and provided general upper and lower bounds [8]. We will show that if $T = \{0, 1, 2, \ldots, m\} - \{k\}$, then for any odd integer k, $\chi_\beta(m, k) = \lceil R(T) \rceil = \lceil \frac{m+k+1}{2} \rceil$ (*i.e.* sharpness for the second inequality in (1).) Exact values of $\chi_\beta(m, k)$ for $k = 2$, and 4, and partial solutions for other even values of k are obtained. We present the proofs by demonstrating periodic optimal colorings with the *smallest* known periods.

The author has recently discussed the results of this article with Gerard Chang and Xuding Zhu. An ensuing collaboration settled the values of $\chi_\beta(m, k)$ for all values of m and k by using different methods [2], which, however, do not guarantee the smallest periods.

2. Connections Between $R(T)$ and $\chi(\mathcal{Z}, D)$

In this section, we show that $R(T)$ is a lower bound of $\chi(\mathcal{Z}, D)$, provided $D = T - \{0\}$. The bound is sharp for a number of families of T-sets including the ones listed below. For any $a, b \in \mathcal{Z}$, let $[a, b]$ denote the set of integers $\{a, a+1, a+2, \ldots, b\}$.

1) *r-initial sets*: $T = [0, r] \cup A$, where A contains no multiple of $(r + 1)$;
2) *k-multiple-of-s sets*: $T = \{0, s, 2s, \ldots, ks\} \cup A$, where $A \subseteq [s, ks]$; and
3) $T = \{0\} \cup [a, b]$.

The T-sets in 1) and 2) above are among the few known T-sets for which the following is always true [4, 16, 21]:

$$sp_T(G) = sp_T(K_{\chi(G)}) \text{ for all graphs } G. \tag{*}$$

If $T = \{0\} \cup [a, b]$, then (*) holds only when b is a multiple of a [18].

Theorem 2.1. *For any given T-set, if $D = T - \{0\}$, then $\lceil R(T) \rceil \leqslant \chi(\mathcal{Z}, D)$.*

Proof. Suppose f is a proper coloring of $G(Z, D)$ with $\chi(Z, D)$ colors. For any n, let $V_n := [0, (n-1) \times \chi(Z, D)]$ be a subset of the vertex set $\mathcal{Z}$, then $|V_n| = (n-1) \times \chi(Z, D) + 1$. Because of the pigeonhole principle there has to be a color c such that there are at least n vertices $\{v_1, v_2, \cdots, v_n\} \subseteq V_n$ which are colored by c. Thus, the numbers $v_1, v_2, \cdots, v_n$ form a T-coloring of K_n. Therefore, one has $\sigma_n \leqslant (n-1) \times \chi(Z, D)$, since $v_i \leqslant (n-1) \times \chi(Z, D)$. It follows $R(T) = \lim_{n\to\infty} \frac{\sigma_n}{n} \leqslant \chi(Z, D)$ and $\lceil R(T) \rceil \leqslant \chi(Z, D)$. □

We assume throughout this article, unless indicated, for any given T-set, $D = T - \{0\}$ and vice versa.

Now, we show that the bound in Theorem 2.1 is sharp for the three families of T-sets introduced at the beginning of this section.

Theorem 2.2. *If T is r-initial, that is, $T = [0, r] \cup A$, where A contains no multiple of $(r+1)$, then $\omega(\mathcal{Z}, D) = R(T) = \chi(\mathcal{Z}, D) = r + 1$.*

Proof. Each collection of $r+1$ consecutive vertices forms a clique, so $\chi(\mathcal{Z}, D) \geqslant \omega(\mathcal{Z}, D) \geqslant r + 1$. Define a periodic coloring $f : \mathcal{Z} \to [0, r]$ by:

$$f(x) = x', \text{ where } x \equiv x' \pmod{r+1},\ 0 \leqslant x' \leqslant r.$$

It is easy to verify that f is a proper coloring. Therefore, $\chi(\mathcal{Z}, D) = \omega(\mathcal{Z}, D) = r + 1$. □

Theorem 2.3. *If T is a k-multiple-of-s set, that is, $T = \{0, s, 2s, \ldots, ks\} \cup A$, where $A \subseteq [s, ks]$, then $\omega(\mathcal{Z}, D) = R(T) = \chi(\mathcal{Z}, D) = k + 1$.*

Proof. Suppose T is a k-multiple-of-s set. Then the set of vertices $\{0, s, 2, 3s, \ldots, ks\}$ forms a clique in the distance graph, so $\chi(\mathcal{Z}, D) \geqslant \omega(\mathcal{Z}, D) \geqslant k + 1$. Define a periodic coloring $f : \mathcal{Z} \to [0, k]$ by:

$$f(x) = \lfloor \frac{i}{s} \rfloor, \text{ where } x \equiv i \pmod{(k+1)s}, 0 \leqslant i \leqslant (k+1)s - 1.$$

It is easy to verify that f is a proper coloring. This implies $\chi(\mathcal{Z}, D) \leqslant k + 1$, hence $\chi(\mathcal{Z}, D) = \omega(\mathcal{Z}, D) = k + 1$. □

Let $T = \{0\} \cup [a, b]$. If b is a multiple of a, then T is a k-multiple-of-s set for which the result has been proved in the theorem above. If b is not a multiple of a, we have the following result which, excluding the $R(T)$ part, was proved in [15] and can be obtained from Theorem 1 in [8]. We include a proof here for completeness.

Theorem 2.4. *If $T = \{0\} \cup [a, b]$, where $b = ak + r$ and $0 < r < a$, then $\omega(\mathcal{Z}, D) = k + 1$ and $\lceil R(T) \rceil = \chi(\mathcal{Z}, D) = k + 2$.*

Proof. Suppose $T = \{0\} \cup [a, b]$, where $b = ak + r$ and $0 < r < a$. In $G(\mathcal{Z}, D)$, it is proved [18] and indeed not difficult to verify that the set of vertices $\{0, a, 2a, \ldots, ka\}$ generates a maximum clique, so $\omega(\mathcal{Z}, D) = k + 1$. It is known [22, 18] that if $T = \{0\} \cup [a, b]$, then $R(T) = \frac{a+b}{a}$. This implies that $\chi(\mathcal{Z}, D) \geqslant k + 2$. Hence, it is enough to find a proper $(k+2)$-coloring for $G(\mathcal{Z}, D)$. Define a coloring $f : \mathcal{Z} \to [0, k+1]$ by,

$$f(x) = \lfloor \frac{i}{a} \rfloor, \text{ where } x \equiv i \pmod{(k+2)a}, 0 \leqslant i \leqslant (k+2)a - 1.$$

It is not difficult to verify that f is a proper coloring. □

It was characterized in [16] that a T-set has the property (*) if and only if $\chi(G_T^n) = \omega(G_T^n)$ for all $n \geqslant 1$. Other known T-sets with the property (*) include *extended k-multiple-of-s sets* [16] and $T = \{0, 1, 3, 5, 7, 9\}$ [17]. An extended k-multiple-of-s-set is constructed from a k-multiple-of-s-set by adding more numbers greater than ks into T so that the equality $\chi(G_T^n) = \omega(G_T^n)$ for all $n \geqslant 1$ will not be violated. By using the same proper coloring defined in the proof of Theorem 2.3, it can be verified that the conclusion of Theorem 2.3 also holds for extended k-multiple-of-s-sets. On the other hand, if $T = \{0, 1, 3, 5, 7, 9\}$, then $\omega(\mathcal{Z}, D) = \chi(\mathcal{Z}, D) = 2$, since D is a subset of odd integers. Therefore, we conclude that for all the known T-sets with property (*), $\omega(\mathcal{Z}, D) = R(T) = \chi(\mathcal{Z}, D)$. Thus, we propose the following conjecture:

Conjecture. *If T satisfies the property (*), then $\omega(\mathcal{Z}, D) = R(T) = \chi(\mathcal{Z}, D)$.*

3 Exact values of $\chi_\beta(m,k)$

This section focuses on the computation of $\chi_\beta(m,k)$ which is the chromatic number of $G(\mathcal{Z},D)$ as $D=[1,m]-\{k\}$. Let $D_{m,k}$ and $T_{m,k}$ denote the sets $D=[1,m]-\{k\}$ and $T=[0,m]-\{k\}$, respectively. We shall first calculate the exact value of $R(T_{m,k})$ which, by Theorem 2.1, is the lower bound of $\chi_\beta(m,k)$. Then, we show that the bound is sharp for many values of m and k including all pairs of integers of m and k where k is odd. Partial results for other even integers k will lead to complete solutions for $k=2$ and 4. Furthermore, all optimal proper colorings presented in this section are periodic with the smallest known periods.

Eggleton, Erdős, and Skilton [8] proved that for any finite D-set, if $G(\mathcal{Z},D)$ is k-colorable, then it has a periodic proper coloring with period at most dk^d, where $d=max\{i : i\in D\}$. As will be shown in this section, the periods for the distance graph $G(\mathcal{Z},D_{m,k})$ could be much smaller than that. The method used in this section provides optimal periodic colorings with very small periods for many values of m and k. For example, we reduce the period for $D=[1,19]-\{8\}$ from 240 (cf. [27]) to 29. (See Example 3.21 at the end of the section.)

Early work about $\chi_\beta(m,k)$ is due to Eggleton, Erdős, and Skilton [8] who settled the case $k=1$: $\chi_\beta(m,1)=\lfloor\frac{1}{2}(m+3)\rfloor$, partially solved the case $k=2$ (see Corollary 3.18 below) and provided the following general bounds:

$$\max\left\{k,\lfloor\frac{1}{2}(\frac{m}{k-1}+1)\rfloor t\right\}\leqslant\chi_\beta(m,k)\leqslant\min\left\{m,\lfloor\frac{1}{2}(\frac{m}{k}+3)\rfloor k\right\}\qquad(2)$$

where $t:=2$ if $k=3$ and $t:=k-2$ if $k\geqslant 4$.

Note that if $m<2k$, then $T=[0,m]-\{k\}$ is an r-initial set with $r=k-1$, so by Theorem 2.2, $\omega(\mathcal{Z},D_{m,k})=R(T_{m,k})=\chi_\beta(m,k)=k$. Hence, throughout this section, we shall assume $m\geqslant 2k$, unless indicated.

Theorem 3.1. *If $m\geqslant 2k$, then $R(T_{m,k})=\frac{m+k+1}{2}$.*

Proof. It is easy to see that the sequence $\sigma=\{\sigma_n\}_{n=1}^{\infty}$ is the following:

$$\sigma=0,k,m+k+1,m+2k+1,2m+2k+2,\dots\dots\dots\dots$$

That is, $\sigma_n=\frac{n-1}{2}(m+k+1)$ if n is odd; and $\sigma_n=\frac{n}{2}(m+k+1)-(m+1)$ otherwise. Hence $R(T)=\lim_{n\to\infty}\frac{\sigma_n}{n}=\lim_{n\to\infty}\frac{m+k+1}{2}$. $\square$

Corollary 3.2. *If $m\geqslant 2k$, then $\chi_\beta(m,k)\geqslant\lceil\frac{m+k+1}{2}\rceil$.*

Proof. The result follows directly from Theorems 2.1 and 3.1. $\square$

Note that the lower bound of $\chi_\beta(m,k)$ in the corollary above improves the one in (2).

The following two lemmas will be used to prove that the inequality in Corollary 3.2 is sharp for some values of m and k, and not for some others.

Lemma 3.3. *If m and k have different parity and $\chi_\beta(m,k) = \frac{m+k+1}{2}$, then any optimal proper coloring of $G(\mathcal{Z}, D_{m,k})$ is periodic with period $m+k+1$.*

Proof. Suppose f is an optimal proper coloring using colors $[1, \frac{m+k+1}{2}]$. Let G be the subgraph of the distance graph induced by the vertex set $[0, m+k]$, $G = G(\mathcal{Z}_{m+k+1}, D_{m,k})$. Obviously, we have $\alpha(G) = 2$, since if $\alpha(G) \geqslant 3$, then $\sigma_3 \leqslant m+k$ contradicting to the proof of Theorem 3.1. Since $\chi_\beta(m,k) = \frac{m+k+1}{2}$, every color in $[1, \frac{m+k+1}{2}]$ has to be assigned to exactly two vertices of G.

Let G' be the subgraph of $G(\mathcal{Z}, D_{m,k})$ induced by the vertex set $[1, m+k+1]$, then $G' \cong G$. Therefore $\alpha(G') = 2$, so each color has to be used by exactly two vertices in G'. This implies $f(m+k+1) = f(0)$. The same argument implies that $f(i) = f(m+k+i+1)$ for any i. This completes the proof. □

Lemma 3.4. *If m and k have different parity, and $\chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil = \frac{m+k+1}{2}$, then $\chi_\beta(m-1,k) = \lceil R(T_{m-1,k}) \rceil = \frac{m+k+1}{2}$.*

Proof. Because $G(\mathcal{Z}, D_{m-1,k})$ is a subgraph of $G(\mathcal{Z}, D_{m,k})$, so $\chi_\beta(m-1,k) \leqslant \chi_\beta(m,k)$. Suppose m and k have different parity and $\chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil$, then we have

$$\lceil R(T_{m-1,k}) \rceil = \lceil \frac{m+k}{2} \rceil = \frac{m+k+1}{2} \leqslant \chi_\beta(m-1,k) \leqslant \chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil = \frac{m+k+1}{2}.$$

Therefore, $\chi_\beta(m-1,k) = \lceil R(T_{m-1,k}) \rceil$. □

We are now at a position to show as k is any odd, and $m \geqslant 2k$, then $\chi_\beta(m,k)$ reaches the lower bound $\lceil R(T_{m,k}) \rceil = \lceil \frac{m+k+1}{2} \rceil$. The simpler format of the proof included here is suggested by Gerard J. Chang through private communication.

Theorem 3.5. *If k is odd, then $\chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil = \lceil \frac{m+k+1}{2} \rceil$.*

Proof. Suppose k is odd. By Lemma 3.4, it suffices to prove the result as m is even.

By Corollary 3.2, it is enough to find a proper coloring for $G(\mathcal{Z}, D_{m,k})$ with $\frac{m+k+1}{2}$ colors. Define a periodic coloring $f : \mathcal{Z} \to [0, \frac{m+k+1}{2} - 1]$ with period $(m+k+1)$ by

$$f(x) = \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even} \\ \frac{i+m+1}{2}, & \text{if } i \text{ is odd and } i < k \\ \frac{i-k}{2}, & \text{if } i \text{ is odd and } i \geqslant k, \end{cases}$$

where $x \equiv i \pmod{m+k+1}$, $0 \leqslant i \leqslant m+k$.

Now we show that f is a proper coloring. Suppose $f(x) = f(y)$, $x \neq y$. Let $x \equiv x' \pmod{m+k+1}$, $y \equiv y' \pmod{m+k+1}$, where $0 \leqslant x', y' \leqslant m+k$.

If x' and y' are both even, then $x' = y'$. This implies $x \equiv y \pmod{m+k+1}$, so x and y are not adjacent.

If x' and y' are both odd, then without loss of generality, we have either $x'-k = y'-k$ or $x'-k = y'+m+1$. For the former case, $x' = y'$; and for the latter case, $x' = y'+m+k+1$. Any of the two cases implies $x \equiv y \pmod{m+k+1}$, hence x and y are not adjacent.

If x' and y' are of different parity, assume x' is even and y' is odd. Then either $x' = y'+m+1$ or $x' = y'-k$, that is, either $x'-y' = m+1$ or $y'-x' = k$. Therefore, we have either $x - y \equiv m + 1 \pmod{m + k + 1}$ or $y - x \equiv k \pmod{m + k + 1}$. In any case, x and y are not adjacent. The proof is complete. □

To solve the case for k even, we let $m = 2nk + r$, $n \in \mathcal{Z}^+$, $0 \leqslant r \leqslant 2k - 1$. First of all, one can make some observations about the clique size of the distance graph. If $r < k$, then the set of vertices $[0, k-1] \cup [2k, 3k-1] \cup [4k, 5k-1] \cup \ldots \cdots \cup [2nk, 2nk + r]$ induces a maximum clique. If $r \geqslant k$, then the set of vertices $[0, k-1] \cup [2k, 3k-1] \cup [4k, 5k-1] \cup \ldots \cdots \cup [2nk, 2nk + k - 1]$ induces a maximum clique. Therefore, we have the following:

Proposition 3.6. *Suppose $m = 2nk + r$, $n \in \mathcal{Z}^+$ and $0 \leqslant r \leqslant 2k - 1$. Then*

$$\omega(\mathcal{Z}, D_{m,k}) = \begin{cases} nk + r + 1, & \text{if } 0 \leqslant r \leqslant k - 1; \\ nk + k, & \text{if } k \leqslant r \leqslant 2k - 1. \end{cases}$$

Lemma 3.7. *If $m = 2nk + r$, $n \in \mathcal{Z}^+$ and $0 \leqslant r \leqslant 2k - 1$, then*

$$\chi_\beta(m, k) \leqslant \begin{cases} nk + k = \frac{m+2k-r}{2}, & \text{if } r < k; \\ nk + r + 1 = \frac{m+r+2}{2}, & \text{if } r \geqslant k. \end{cases}$$

Proof. We define a periodic coloring on $G(\mathcal{Z}, D_{m,k})$ with period $m + k + 1$ by first partitioning the vertices $[0, m + k]$ into two parts, $A = [0, 2nk - 1]$ and $B = [2nk, 2nk + k + r]$. Then $|A| = 2nk$ and $|B| = k + r + 1$.

Now, color the vertices in A with nk colors as follows. Assign the first k colors to the first $2k$ vertices by $f(x + k) = f(x) = x$, $0 \leqslant x \leqslant k - 1$; then a different set of k colors to the next $2k$ vertices, and so on, until all vertices in A are colored. Similarly, if $r < k$, assign k colors to vertices in B by: the first k vertices use k colors and for any of the remaining $r + 1$ vertices x, let $f(x) = f(x - k)$. If $r \geqslant k$, assign k colors to the first $2k$ vertices in B, and assign a new color to each of the remaining $r - k + 1$ vertices in B. Then for either of these two cases, repeat this coloring to all $\mathcal{Z}$ periodically, *i.e.* $f(x) = f(y)$ if $x \equiv y \pmod{m + k + 1}$.

It is easy to verify that f is indeed a proper coloring. If $f(x) = f(y)$, then $|x - y|$ is either k or at least $m + 1$. The total number of colors used by f is $nk + k = \frac{m+2k-r}{2}$ if $r < k$; or $(nk + k) + (r - k + 1) = nk + r + 1 = \frac{m+r+2}{2}$ if $r \geqslant k$. The proof is complete. □

Note that the upper bounds in the lemma above are either better than or equal to the one in (2).

With the following five theorems or corollaries, we show the bound in Corollary 3.2 is also sharp for some even numbers k and special values of m.

Theorem 3.8. *If $m \equiv k \pmod{2k}$ and $m \geqslant 2k$, then $\chi_\beta(m, k) = \lceil \frac{m+k+1}{2} \rceil$.*

Proof. Let $m = 2nk + k$, then $\lceil \frac{m+k+1}{2} \rceil = nk + k + 1$. By Corollary 3.2 and Lemma 3.7, we obtain $\chi_\beta = \lceil \frac{m+k+1}{2} \rceil = nk + k + 1$. □

Theorem 3.9. *If $m \equiv k-1 \pmod{2k}$ and $m \geqslant 2k$, then $\chi_\beta(m,k) = \lceil \frac{m+k+1}{2} \rceil$.*

Proof. Let $m = 2nk + k - 1$, then $\lceil \frac{m+k+1}{2} \rceil = nk + k$. By Corollary 3.2 and Lemma 3.7, we obtain $\chi_\beta(m,k) = \lceil \frac{m+k+1}{2} \rceil = nk + k$. □

Corollary 3.10. *If $m \equiv k-2 \pmod{2k}$ and $m \geqslant 2k$, then $\chi_\beta(m,k) = \lceil \frac{m+k+1}{2} \rceil$.*

Proof. By Theorem 3.5, we only have to show the equality when k is even. Then $k-1$ is odd, by Lemma 3.4 and Theorem 3.9, the proof is complete. □

In the following two theorems, we prove $\chi_\beta(m,k) = \lceil \frac{m+k+1}{2} \rceil$ for some special values of m and k by presenting periodic proper colorings with period $m+k+1$. To check that the colorings are proper is routine. We strongly recommend the reader to look at special values of m and k as examples.

Theorem 3.11. *If k is even, $m \equiv 0 \pmod{2k}$ and $m \geqslant 2k$, then $\chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil = \frac{m+k+2}{2}$.*

Proof. Let $m = 2nk$, then $\lceil R(T_{m,k}) \rceil = nk + \frac{k}{2} + 1$, and $\omega(\mathcal{Z}, D_{m,k}) = nk + 1$. Let $G = G(Z_{m+k+1}, D_{m,k})$. Define the periodic coloring f with period $m+k+1$ as follows. First, color the maximum clique in G by:

$$\begin{array}{llll} f(0)=0, & f(1)=1, & \ldots\ldots, & f(k-1)=k-1, \\ f(2k)=k, & f(2k+1)=k+1, & \ldots\ldots, & f(3k-1)=2k-1, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ f(2nk)=nk. & & & \end{array}$$

Secondly, let $f(2nk+1) = nk+1$, $f(2nk+3) = nk+2$, $\ldots$, and $f(2nk+k-1) = nk + \frac{k}{2}$. Finally, color the remaining vertices in G by:

$$f(x) = \begin{cases} f(x-k), & \text{if } x \text{ is even and } x \leqslant 2nk-2; \\ f(x-m-1), & \text{if } x \text{ is even and } x \geqslant 2nk+2; \\ f(x+k), & \text{if } x \text{ is odd and } x \leqslant 2nk-1. \end{cases}$$

Then repeat this coloring to all integers $\mathcal{Z}$ periodically. It is not hard to check that if $f(x) = f(y)$, then $|x-y|$ is either k or at least $m+1$. Therefore, f is a proper coloring. The proof is complete. □

Theorem 3.12. *If k is even, $k \geqslant 4$, $m \equiv 2k-2 \pmod{2k}$ and $m \geqslant 2k$, then $\chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil = \frac{m+k+2}{2}$.*

Proof. Let $m = 2nk + 2k - 2$, then $\lceil R(T_{m,k}) \rceil = nk + k + \frac{k}{2}$. By Prop. 3.6, $\omega(\mathcal{Z}, D_{m,k}) = nk + k$. Let $G = G(\mathcal{Z}_{m+k+1}, D_{m,k})$. Define a periodic coloring f with period $m+k+1$ as follows. First, color the maximum clique in G by:

$$\begin{array}{llll} f(0)=0, & f(1)=1, & \ldots\ldots, & f(k-1)=k-1, \\ f(2k)=k, & f(2k+1)=k+1, & \ldots\ldots, & f(3k-1)=2k-1, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ f(2nk)=nk, & f(2nk+1)=nk+1, & \ldots\ldots & f(2nk+k-1)=nk+k-1. \end{array}$$

Secondly, color any vertex $x \geqslant 2nk + 2k$ in G by:

$$f(x) = \begin{cases} nk + k + \frac{i-1}{2}, & \text{if } x \text{ is odd and } x = 2nk + 2k + i,\ 1 \leqslant i \leqslant k - 3; \\ f(x - m - 1), & \text{if } x \text{ is even and } 2nk + 2k \leqslant x \leqslant 2nk + 3k - 4; \\ nk + k + \frac{k}{2} - 1, & \text{if } x = 2nk + 3k - 2. \end{cases}$$

Finally, for any remaining vertex x in G, let $x = ak+i$, $1 \leqslant a \leqslant 2n+1$ and $0 \leqslant i \leqslant k-1$. We color x according to the following:

$$f(x) = \begin{cases} f(x - k), & \text{if } i = k - 1; \text{ or } x \text{ is even and } i \leqslant k - 2; \\ f(x + k), & \text{if } x \text{ is odd and } i \neq k - 1. \end{cases}$$

Then extend the coloring f periodically to $\mathcal{Z}$. It is routine to verify that the extended f is a proper coloring for $G(Z, D)$. □

Now we show that the bound of $\chi_\beta(m, k)$ in Corollary 3.2 is not always tight.

Theorem 3.13. *If k is even, $m \equiv k + 1 \pmod{2k}$ and $m \geqslant 2k$, then $\chi_\beta(m, k) = \lceil R(T_{m,k}) \rceil + 1 = \frac{m+k+3}{2}$.*

Proof. Let $m = 2nk + k + 1$, then $\lceil R(T_{m,k}) \rceil = nk + k + 1$. By Prop. 3.6, $\omega(\mathcal{Z}, D_{m,k}) = nk + k$. Suppose $\chi_\beta(m, k) = \lceil R(T_{m,k}) \rceil = nk + k + 1$. Let f be an optimal proper coloring with colors $[0, nk+k]$ and let $G = G(\mathcal{Z}_{m+k+1}, D_{m,k})$. Without loss of generality, f assigns colors $[0, nk + k - 1]$ to the maximum clique in G by

$$\begin{array}{cccc} f(0) = 0, & f(1) = 1, & \ldots\ldots & f(k-1) = k-1, \\ f(2k) = k, & f(2k+1) = k+1, & \ldots\ldots, & f(3k-1) = 2k-1, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ f(2nk) = nk, & f(2nk+1) = nk+1, & \ldots\ldots, & f(2nk+k-1) = nk+k-1. \end{array}$$

By Lemma 3.3, each color is used exactly twice in G and f is periodic with period $m + k + 1$, so $f(m + k + 1) = 0$, $f(m + k + 2) = 1$, etc. Then one can find each remaining vertex x in G a set of "potential" colors $p(x)$ which are the possible colors from $[0, nk+k-1]$ that can be assigned to x. For instance, $p(k) = \{0, k\}$, $p(2nk + k + 2) = \{0, nk + 2\}$, and $p(m + k) = \{k - 1\}$. Suppose y is a vertex in the maximum clique and $f(y) = c \in [0, nk + k - 1]$, then c is a potential color for exactly two other vertices in G, $y - k \pmod{m + k + 1}$ and $y + k \pmod{m + k + 1}$.

A vertex $v \in G$ is called an even (or odd) vertex if v is even (or odd). The union of the potential colors of all even (or odd, respectively) vertices is the set of all even (or odd, respectively) numbers from $[0, nk + k - 1]$. Now, the new color $nk + k$ must be received either by two odd vertices or two even vertices. Suppose $nk + k$ is assigned to two even vertices. The number of odd vertices in G is $kn + k + 1$ which is odd, so it is impossible to use each odd color twice. A similar contradiction arrives when $nk + k$ is assigned to two odd vertices. Thus, $\chi_\beta(m, k) > \lceil R(T_{m,k}) \rceil$.

Define a periodic coloring $f : \mathcal{Z} \to [0, nk+k+1]$ by:

$$f(x) = \lfloor \frac{a}{2} \rfloor k + i.$$

where $y \equiv x \pmod{m+k+1}$, $0 \leqslant y \leqslant m+k$, $y = ak+i$, $0 \leqslant i \leqslant k-1$.

It is not hard to check that f is a proper coloring with period $m+k+1$. The proof is complete. □

The argument used in the proof above can not be extended to all odd m and even k (see Theorem 3.16 below). However, it works when $\frac{m+k+1}{2}$ is odd, so we have the following result:

Theorem 3.14. *If k is even, $m+k+1 \equiv 2 \pmod 4$ and $m \geqslant 2k$, then $\chi_\beta(m,k) > \frac{m+k+1}{2}$.*

Theorem 3.15. *If k is even, $k \geqslant 4$, $m \equiv k-3 \pmod{2k}$ and $m \geqslant 2k$, then $\chi_\beta(m,k) = \frac{m+k+3}{2}$.*

Proof. Let $m = 2nk+k-3$, then $m+k+1 \equiv 2 \pmod 4$, by Theorem 3.14, $\chi_\beta(m,k) > \frac{m+k+1}{2}$. According to Corollary 3.10, $\chi_\beta(m,k) \leqslant \chi_\beta(m+1,k) = \frac{m+k+3}{2}$. □

Theorem 3.16. *If $m = 12n+1$, then $\chi_\beta(m,6) = \lceil R(T_{m,6}) \rceil = 6n+4$.*

Proof. First, let $m = 13$, then $\omega(\mathcal{Z}, D_{13,6}) = 8$ and $\lceil R(T_{13,6}) \rceil = 10$. Define a periodic 10-coloring f with period 20 by the following sequence $f(0)$, $f(1)$,... $f(19)$:

$$\boxed{1,2,3,4,5,6,} \quad \underline{1}, \underline{2}, \overline{9}, \overline{10}, \underline{5}, \underline{6}, \quad \boxed{7,8,} \quad \mathbf{9}, \mathbf{10}, \overline{3}, \overline{4}, \underline{7}, \underline{8}$$

The numbers within the two boxes above correspond to the maximum clique. A bold number represents a new color. An underlined number is the same color as the 6-th position preceding it; and an overlined number is the same color as the 6-th position following it, circulantly (mod 20).

It is easy to see that f is a proper coloring. The pattern used in f can be extended to $m = 12n+1$ as follows. If $m = 12n+1$ with $n > 1$, then the maximum clique in $G = (\mathcal{Z}_{m+7}, D_{m,6})$ consists of $n+1$ blocks with only two vertices in the last block.

Now, color the maximum clique with sequential colors. Then, for vertices after the last block, use the same underline-overline pattern as the above by replacing 9 and 10 by two new colors. For every set of 6 vertices between two blocks, follow the underline-overline pattern used between the two boxes in the above. Then, we get a proper periodic coloring for $m = 12n+1$. □

Theorem 3.17. *If k is even and $m \equiv 2k-1 \pmod{2k}$, then $\chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil + 1 = \frac{m+k+3}{2}$.*

Proof. Let $m = 2nk+2k-1$, then $\omega(\mathcal{Z}, D_{m,k}) = nk+k$ and $\lceil R(T_{m,k}) \rceil = nk+k+\frac{k}{2}$. To show $\chi_\beta(m,k) > \lceil R(T_{m,k}) \rceil$, we use an argument similar to the proof of Theorem 3.13. Suppose $\chi_\beta(m,k) = \lceil R(T_{m,k}) \rceil$ and let f be an optimal proper coloring. By Lemma 3.3,

f is periodic with period $m+k+1$. Without loss of generality, f colors the maximum clique in $G = G(\mathcal{Z}_{m+k+1}, D_{m,k})$ by:

$$\begin{array}{cccc} f(0)=0, & f(1)=1, & \ldots\ldots, & f(k-1)=k-1, \\ f(2k)=k, & f(2k+1)=k+1, & \ldots\ldots, & f(3k-1)=2k-1, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ f(2nk)=nk, & f(2nk+1)=nk+1, & \ldots\ldots, & f(2nk+k-1)=nk+k-1. \end{array}$$

By Lemma 3.3, f assigns each color to exactly two vertices in G. Therefore, one can find the set of potential colors $p(x)$ to each of the remaining vertices in G.

Then, the new color $N = nk$ must be assigned to exactly two vertices, x_1, $x_2 \in [2nk+k, 2nk+3k-1]$ such that $|x_1 - x_2| = k$. Suppose $f(2nk+k) = f(2nk+2k) = nk$, then the set of vertices that are either colored by or have the potential colors $\{0, k, 2k, \ldots, nk\}$ is $A = \{0, k, 2k, 3k, \ldots, 2nk, 2nk+k, 2nk+2k\}$. Since $|A|$ is odd, it is impossible to use each color exactly twice. Similarly, one can show that it is impossible to have $f(2nk+k+i) = f(2nk+2k+i) = nk$. Therefore, $\chi_\beta(m,k) > \frac{m+k+1}{2}$.

By Theorem 3.11, $\chi_\beta(m,k) \leqslant \chi_\beta(m+1,k) = \frac{m+k+3}{2}$. The proof is complete. □

Corollary 3.18. *Suppose $m = 4n + r$, $0 \leqslant r \leqslant 3$, then*

$$\chi_\beta(m,2) = \begin{cases} \lceil \frac{m+3}{2} \rceil, & \text{if } r = 0,1,2; \\ \frac{m+5}{2}, & \text{if } r = 3. \end{cases}$$

Proof. For $r = 0, 1$, and 2, the results follow from Theorems 3.11, 3.9, and 3.8, respectively. For $r = 3$, the result is true by Theorem 3.13. □

Note that the cases for $r = 0, 1$ and 2 in the theorem above were first proved by Eggleton, Erdős, and Skilton [8].

Corollary 3.19. *Suppose $m = 8n + r$, $0 \leqslant r \leqslant 7$, then*

$$\chi_\beta(m,4) = \begin{cases} \lceil \frac{m+5}{2} \rceil, & \text{if } r = 0,2,3,4,6; \\ \frac{m+7}{2}, & \text{if } r = 1,5,7. \end{cases}$$

Proof. For $r = 0, 2, 3, 4$ and 6, the results follow from Theorem 3.11, Corollary 3.10, and Theorems 3.9, 3.8, and 3.12, respectively. For $r = 1, 5$ and 7, the claims result from Theorems 3.15, 3.13, and 3.17, respectively. □

Corollary 3.20. *Suppose $m = 12n + r$, $0 \leqslant r \leqslant 11$, then*

$$\chi_\beta(m,6) = \begin{cases} \lceil \frac{m+7}{2} \rceil, & \text{if } r = 0,1,4,5,6,8,9,10; \\ \frac{m+9}{2}, & \text{if } r = 3,7,11; \\ \leqslant \frac{m+10}{2}, & \text{if } r = 2. \end{cases}$$

Proof. The cases for $r = 0, 1, 4, 5, 6$, and 10 can be obtained from Theorems 3.11, 3.16, Corollary 3.10, and Theorems 3.9, 3.8, and 3.12, respectively. The results for $r = 3, 7$ and 11 follow from Theorems 3.15, 3.13 and 3.17, respectively.

If $r = 9$, consider $m = 21$. Define a periodic coloring with period 28 by the following sequence $f(0)$, $f(2)$, ..., $f(27)$:

$$\boxed{1, 2, 3, 4, 5, 6,} \quad \underline{1}, \overline{8}, \overline{9}, \underline{4}, \underline{5}, \overline{12}, \quad \boxed{7, 8, 9, 10, 11, 12,} \quad \underline{7}, \mathbf{1}3, \mathbf{1}4, \underline{10}, \underline{11}, \overline{2}, \overline{3}, \underline{13}, \underline{14}, \overline{6}$$

The notations used above are the same as the ones as the coloring given in Theorem 3.16. It is easy to check that f is a proper coloring and the pattern used in f can be extended to the case $m = 12n + 9$.

The case for $r = 8$ follows from Lemma 3.4 and the result as $r = 9$.

For $r = 2$, $\lceil R(T_{m,k}) \rceil = \frac{m+8}{2}$. By the "maximum clique and potential colors" method used in Theorem 3.13, it can be shown that there is no periodic proper $(\frac{m+8}{2})$-coloring with period $m + 7$, however, this does not imply $\chi_\beta(m, 6) > \frac{m+8}{2}$ (since Lemma 3.3 does not imply to this case.) At least, we find a periodic $(\frac{m+10}{2})$-coloring with period $m + 7$ for this case. We demonstrate such a coloring f when $m = 14$ by the following sequence, $f(0)$, $f(1)$, ..., $f(20)$:

$$\boxed{1, 2, 3, 4, 5, 6,} \quad \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \quad \boxed{7, 8, 9,} \quad \mathbf{1}0, \mathbf{1}1, \mathbf{1}2, \underline{7}, \underline{8}, \underline{9}.$$

The notations above are the same as the ones used in Theorem 3.16. It is easy to see that f is a proper coloring and the pattern can be extended to $m = 12n + 2$. $\square$

The "maximum clique and potential colors" method of finding periodic colorings with small periods used in this section can also be extended to other values of m and k. The following is an example:

Example 3.21. $\chi(19, 8) = 15$, *and there exists a periodic proper* 15*-coloring with period 29.*

An argument similar to the proof of Theorem 3.17 implies that $\chi(19, 8) > 14$. The following sequence shows a periodic proper 15-coloring with period 29.

$$\boxed{1, 2, 3, 4, 5, 6, 7, 8,} \quad \underline{1}, \overline{10}, \overline{11}, \underline{4}, \overline{13}, \overline{14}, \underline{7}, \underline{8}, \quad \boxed{9, 10, 11, 12,} \quad \mathbf{1}3, \mathbf{1}4, \overline{2}, \overline{3}, \underline{9}, \overline{5}, \overline{6}, \underline{12}, \mathbf{1}5. \quad \square$$

Concluding Remark. We have learned that $\chi_\beta(m, k) = R(T_{m,k}) = k$ if $m < 2k$. Suppose $m \geqslant 2k$, then $\chi_\beta(m, k) = \lceil \frac{m+k+1}{2} \rceil$ if k is odd. If k is even, Corollaries 3.18 and 3.19 settled the cases $k = 2$ or 4, respectively. In addition to partial solutions for the case $k = 6$ (Corollary 3.20,) by the results obtained in this section, we conclude the following for some other cases:

Suppose k is even, $k \geqslant 6$, $m = 2nk + r$, $0 \leqslant r \leqslant 2k - 1$, and $m \geqslant 2k$, then we have

$$\chi_\beta(m, k) = \begin{cases} \lceil \frac{m+k+1}{2} \rceil, & \text{if } r = 0, k-2, k-1, k, 2k-2; \\ \lceil \frac{m+k+1}{2} \rceil + 1, & \text{if } r = k-3, k+1, 2k-1. \end{cases}$$

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References

1. D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. of Comb. Theory, Series A **14** (1973), 281-287.
2. G. J. Chang, D. D.-F. Liu and X. Zhu, *Distance graphs and T-coloring*, manuscript (1997).
3. J. J. Chen, G. J. Chang and K.C. Huang, *Integral distance graphs*, Journal of Graph Theory (to appear).
4. M. B. Cozzens and F. S. Roberts, *T-colorings of graphs and the channel assignment problem*, Congressus Numerantium **35** (1982), 191-208.
5. M. B. Cozzens and F. S. Roberts, *Greedy algorithms for T-colorings of complete graphs and the meaningfulness of conclusions about them*, J. Comb. Inform. Syst. Sci. **16** (1991), 286-299.
6. W. Deuber and X. Zhu, *The chromatic number of distance graphs*, Disc. Math. **165/166** (1997), 195-204.
7. Eggleton, *New results on 3-chromatic prime distance graphs*, Ars Comb. **26B** (1988), 153-180.
8. R. B. Eggleton, P. Erdős, and D. S. Skilton, *Colouring the real line*, Journal of Combinatorial Theory Series B **39** (1985), 86-100.
9. R. B. Eggleton, P. Erdős, and D. S. Skilton, *Research problem 77*, Disc. Math. **58** (1986), 323.
10. R. B. Eggleton, P. Erdős, and D. S. Skilton, *Colouring prime distance graphs*, Graph and Combinatorics **32** (1990), 17-32.
11. J. R. Griggs and D. D.-F. Liu, *The channel assignment problem for mutually adjacent sites*, Journal of Combinatorial Theory, Series A **68** (1994), 169 - 183.
12. H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial geometry in the plane*, Holt Rinehart and Winston, New York (1964).
13. Haralambis, *Sets of integers with missing differences*, Journal of Combinatorial Theory, Series A **23** (1977), 22-23.
14. W. K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE **68** (1980), 1497-1514.
15. A. Kemnitz and H. Kolberg, *Coloring of integer distance graphs*, Tech. report, Institute Für Mathematik, Technische Universität Braunschweig (1996).
16. D. D.-F. Liu, *T-coloring of graphs*, Discrete Math. **101** (1992), 203 - 212.
17. D. D.-F. Liu, *On a conjecture of T-colorings*, Congressus Numerantium **103** (1994), 27-31.
18. D. D.-F. Liu, *T-graphs and the channel assignment problem*, Discrete Math. **161** (1996), 197-205.
19. L. Moser and W. Moser, *Solution to problem 10*, Candad. Math. Bull. **4** (1961), 187-189.
20. J. H. Rabinowitz and V. K. Proulx, *An asymptotic approach to the channel assignment problem*, SIAM Journal on Algebraic and Discrete Methods **6** (1985), 507-518.
21. A. Raychaudhuri, *Further results on T-coloring and frequency assignment problems*, SIAM of Discrete Math **7** (1994), 605–613.
22. B. Tesman, *T-colorings, List T-colorings and set T-colorings of graphs*, Ph.D. Dissertation, Department of Mathematics, Rutgers University (1989).
23. Voigt, *Coloring of distance graphs*, Ars Comb. (to appear.).
24. M. Voigt and H. Walther, *Chromatic number of prime distance graphs*, Disc. Appl. Math **51** (1994), 197-209.
25. X. Zhu, *Distance graphs on the real line*, manuscript (1996).
26. X. Zhu, *Non-integral distance graphs on the real line*, manuscript (1996).
27. X. Zhu, *Pattern periodic coloring of distance graphs*, manuscript (1997).