Circular Chromatic Numbers of Some Reduced Kneser Graphs

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Abstract

The vertex set of the reduced Kneser graph $\mathrm{KG}_2(m,2)$ consists of all pairs $\{a,b\}$ such that $a,b \in \{1,2,\ldots,m\}$ and $2 \le |a-b| \le m-2$. Two vertices are defined to be adjacent if they are disjoint. We prove that, if $m \ge 4$ and $m \ne 5$, then the circular chromatic number of $\mathrm{KG}_2(m,2)$ is equal to $m-2$, its ordinary chromatic number.

Keywords: *circular chromatic number, Kneser graph, reduced Kneser graph*

1 Introduction

Given positive integers k and d, $k \ge 2d$, a (k,d)-*coloring* of a graph G is a mapping ϕ from the vertex set $V(G)$ to the set $\{0,1,\ldots,k-1\}$ such that $d \le$

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$|\,\phi(x) - \phi(y)\,| \leq k - d$ whenever x and y are adjacent vertices. The *circular chromatic number* $\chi_c(G)$ is defined to be the infimum of k/d such that G admits a (k, d)-coloring. Vince [7] first introduced this notion of colorability, named it the *star chromatic number*, and proved that the infimum can be attained by a minimum. It is also known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, where $\chi(G)$ is the ordinary chromatic number of G. Hence $\chi(G) = \lceil \chi_c(G) \rceil$. From this point of view, the circular chromatic number can be regarded as a refinement of the ordinary chromatic number.

Circular chromatic numbers have been studied intensively in recent years. Zhu [9] provides a comprehensive survey of this area, in which over one hundred references are listed.

One section of Zhu's survey concentrates on graphs whose circular chromatic numbers equal chromatic numbers. Among such graphs, one group of conspicuous examples includes Kneser graphs of particular parameters. For $m \geq 2n \geq 2$, the *Kneser graph* $\mathrm{KG}(m, n)$ has the vertex set of all n-subsets of the set $[m] = \{1, 2, \ldots, m\}$. Two vertices are defined to be adjacent in $\mathrm{KG}(m, n)$ if they have empty intersection as subsets.

It was conjectured by Kneser [4] in 1955 and proved by Lovász [5] in 1978 that $\chi(\mathrm{KG}(m, n)) = m - 2n + 2$. The proof was a celebrating success because it employed tools from algebraic topology. Recently, Johnson, Holroyd, and Stahl [3] have proved that $\chi_c(\mathrm{KG}(2n+1, n)) = 3$, $\chi_c(\mathrm{KG}(2n+2, n)) = 4$, and $\chi_c(\mathrm{KG}(m, 2)) = m - 2$. Each of these circular chromatic numbers attains its upper bound. In view of these results, they further proposed the following.

Conjecture 1 *For every Kneser graph* $\mathrm{KG}(m, n)$*, we have* $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$.

A subset S of $[m]$ is said to be 2-*stable* if $2 \leq |x - y| \leq m - 2$ for distinct elements x and y of S. The *reduced Kneser graph* $\mathrm{KG}_2(m,n)$ is the subgraph of $\mathrm{KG}(m,n)$ induced by all 2-stable n-subsets. It was proved by Schrijver [6] that $\chi(\mathrm{KG}_2(m,n)) = \chi(\mathrm{KG}(m,n))$ and every subgraph induced by a proper subset of $V(\mathrm{KG}_2(m,n))$ has a smaller chromatic number, i.e., $\mathrm{KG}_2(m,n))$ is vertex-critical.

The main focus of this paper is to show that the circular chromatic number of the reduced Kneser graph $\mathrm{KG}_2(m,2)$ also attains its upper bound if $m \geq 4$ and $m \neq 5$. We first re-prove the result $\chi(\mathrm{KG}(m,2)) = m-2$ by a simpler and straightforward method. Our main result can be established using a refined version of this method.

2 Lemmas

A (k,d)-*partition* of a graph G is a partition $(V_0, V_1, \ldots, V_{k-1})$ of its vertex set $V(G)$ such that $V_i \cup V_{i+1} \cup \cdots \cup V_{i+d-1}$ is an independent set of G for every i, $0 \leq i \leq k-1$, where indices are added modulo k and any V_i is allowed to be empty. The parts V_i's are also called *color classes* and two of them are said to be *consecutive* or *adjacent* if their indices differ by 1 modulo k. The color classes of a (k,d)-partition are simply the color classes of a (k,d)-coloring. The following observation first appeared in Fan [1].

Lemma 2 *A graph G has a (k,d)-coloring if and only if it has a (k,d)-partition.*

The next lemma appeared in Zhu [8].

Lemma 3 *If ϕ is a (k,d)-coloring of G and $\chi_c(G) = k/d$, where* $\gcd(k,d) = 1$*, then ϕ is a surjection.*

In terms of (k, d)-partitions, Fan [1] translated this lemma into the following form.

Lemma 4 *Let* $(V_0, V_1, \ldots, V_{k-1})$ *be a* (k, d)*-partition of* G*, where* $\gcd(k, d) = 1$*. If* $\chi_c(G) = k/d$*, then every* V_i *is nonempty.*

Lemma 5 *Let* $(V_0, V_1, \ldots, V_{k-1})$ *be a* (k, d)*-partition of* G *and* $\chi_c(G) = k/d$*, where* $\gcd(k, d) = 1$*. Then for every* i*,* $0 \leq i \leq k-1$*, there are vertices* x *in* V_i *and* y *in* V_{i+d} *such that* x *and* y *are adjacent.*

The last lemma is an observation made by Hajiabolhassan and Zhu [2]. If no vertex in V_i is adjacent to a vertex in V_{i+d}, then we can construct a new (k, d)-partition by merging V_{i+d-1} and V_{i+d} into a new color class V'_{i+d-1} and let the adjacent color class V'_{i+d} be empty. This would imply $\chi_c(G) < k/d$ by Lemma 4.

3 Main Result

Theorem 6 *If the integer* $m \geq 4$*, then* $\chi_c(\mathrm{KG}(m, 2)) = m - 2$*.*

Proof. Suppose to the contrary that $\chi_c(\mathrm{KG}(m, 2)) = k/d < m - 2$, where $\gcd(k, d) = 1$ and $d \geq 2$. Let $(V_0, V_1, \ldots, V_{k-1})$ be a (k, d)-partition of $\mathrm{KG}(m, 2)$ with nonempty color classes V_i's.

Case 1. For some i, $|V_i| \geq 2$.

Without loss of generality, let $\{1, 2\}$ and $\{1, 3\}$ belong to V_i. By Lemma 5, there are vertices x in V_{i-1} and y in V_{i+d-1} such that x and y are adjacent. Since both x and y are adjacent to neither $\{1, 2\}$ nor $\{1, 3\}$, the only vertices of $\mathrm{KG}(m, 2)$ that could be chosen as x and y are $\{1, 4\}, \{1, 5\}, \ldots, \{1, m\}$ and $\{2, 3\}$. However, $\{2, 3\}$ is adjacent to every vertex of the independent set $\{\{1, 4\}, \{1, 5\}, \ldots, \{1, m\}\}$. Therefore one of x and y must be $\{2, 3\}$.

If $d > 2$, then $\{2, 3\}$ is adjacent to all vertices of V_{i+1}. This adjacency contradicts the defining properties of a (k, d)-coloring. It follows that $d = 2$ and at least one of V_{i-1} and V_{i+1} is a singleton.

Next we claim that $|V_j| \leq 2$ for any j. Let $\{a, b\}$ belong to V_{j-1} and $\{c, d\}$ belong to V_{j+1} such that $\{a, b\}$ and $\{c, d\}$ are adjacent. Thus a, b, c, and d are distinct numbers. The vertices of $\mathrm{KG}(m, 2)$ that are adjacent to neither $\{a, b\}$ nor $\{c, d\}$ belong to $\{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$, which consists of two independent edges of $\mathrm{KG}(m, 2)$. Thus the independent set V_j contains at most two vertices.

We conclude that $2k > |V(\mathrm{KG}(m, 2))| = m(m-1)/2$. Now the fact $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ implies that $\chi_c(\mathrm{KG}(m, 2)) = k/2 = m - 2 - \frac{1}{2}$. It follows that $2(2m-5) > m(m-1)/2$. However, no integer $m \geq 4$ satisfies this inequality.

Case 2. For all i, $|V_i| = 1$.

Suppose that $d \geq 4$. We may suppose that V_0 consists of the unique vertex $\{1, 2\}$ and the unique vertex of V_1 also contains the number 1. Moving along increasing indices, we finally reach some j such that $V_j = \{\{1, a\}\}$, $V_{j+1} = \{\{1, b\}\}$, and $V_{j+2} = \{\{a, b\}\}$. This would force V_{j+3} to be empty. Hence $d \leq 3$.

Note that $k = |V(\mathrm{KG}(m, 2))|$ in this case. If $d = 2$, then $2m - 5 = k = m(m-1)/2$, i.e., $m^2 - 5m + 10 = 0$. No integer $m \geq 4$ satisfies this identity. If $d = 3$, then we have two possibilities: (i) $k/3 = m - 2 - \frac{1}{3}$ or (ii) $k/3 = m - 2 - \frac{2}{3}$. We can derive the identities $m^2 - 7m + 14 = 0$ for (i) and $m^2 - 7m + 16 = 0$ for (ii). Both identities have no solutions for integers $m \geq 4$. $\square$

The circular chromatic numbers of reduced Kneser graphs may be smaller than their chromatic numbers. For instance, the graph $\mathrm{KG}_2(2n+1, n)$

is an odd cycle $C_{2n+1} : x_0, x_1, \ldots, x_{2n}, x_0$, where $x_i = \{1+i, 3+i, \ldots, 2n-1+i\}$ (additions modulo $2n+1$) for $i = 0, 1, \ldots, 2n$. However, it is well-known that $\chi_c(C_{2n+1}) = 2 + \frac{1}{n} < 3 = \chi(C_{2n+1})$ when $n > 1$.

Theorem 7 *If the integer $m \geq 4$ and $m \neq 5$, then $\chi_c(\mathrm{KG}_2(m, 2)) = m - 2$.*

Proof. The graph $\mathrm{KG}_2(4, 2)$ consists of three independent edges and trivially $\chi_c(\mathrm{KG}_2(4, 2)) = 2$.

Assume that $m \geq 6$. Suppose to the contrary that $\chi_c(\mathrm{KG}_2(m, 2)) = k/d < m - 2$, where $\gcd(k, d) = 1$ and $d \geq 2$. Let $(V_0, V_1, \ldots, V_{k-1})$ be a (k, d)-partition of $\mathrm{KG}_2(m, 2)$ with non-empty color classes.

Case 1. For some i, $|V_i| \geq 2$.

We first make the following observation.

$$\text{If } V_i = \{\{x, y\}, \{x, z\}\}, \text{ then } \{y, z\} \text{ is a 2-stable set.} \qquad (*)$$

For otherwise, all the vertices that are adjacent to neither $\{x, y\}$ nor $\{x, z\}$ would contain the number x, hence form an independent set. By Lemma 5, there are vertices u in V_{i-1} and w in V_{i+d-1} such that u and w are adjacent. Since u and w are adjacent to neither $\{x, y\}$ nor $\{x, z\}$, we have obtained a contradiction.

The same argument for Case 1 of Theorem 6 can be used to show that $d = 2$ and at least one of V_{i-1} and V_{i+1} is a singleton. Furthermore, the argument also shows that $|V_j| \leq 2$ for all j.

Now let p denote the number of color classes of size 2. Since no three consecutive color classes are of size 2, we have $p \leq 2k/3$. It follows that $|V(\mathrm{KG}_2(m, 2))| = \binom{m}{2} - m = m(m-3)/2 = 2p + (k - p) \leq k + (2k/3)$. Substituting $k = 2m - 5$ into this inequality, we obtain $3m^2 - 29m + 50 \leq 0$ which can be satisfied only by $m = 6$ and 7.

Assume that $m = 6$ and there is a $(7,2)$-partition $(V_0, V_1, \ldots, V_6)$ of $\mathrm{KG}_2(6,2)$. Because $|V(\mathrm{KG}_2(6,2))| = 9$, at least one color class is of size 2. Also note that, for each $x \in [6]$, there are exactly three vertices in $\mathrm{KG}_2(6,2)$ that contain x: $\{x, x+2\}$, $\{x, x+3\}$, $\{x, x+4\}$ (additions modulo 6). Hence we may assume that $V_0 = \{\{1,3\},\{1,5\}\}$ by $(*)$. Then V_6 and V_1 are singletons and $V_6 \cup V_1 = \{\{3,5\},\{1,4\}\}$. Each vertex of the path P : $\{2,4\}$, $\{3,6\}$, $\{2,5\}$, $\{4,6\}$ is adjacent to either $\{1,3\}$ or $\{1,5\}$. By $(*)$, all the four vertices of P belong to at least three distinct color classes among V_j, $2 \leq j \leq 5$. Since two consecutive vertices on P cannot occur in the same or consecutive color classes, P starts from V_3, then successively moves to V_5, V_2, and terminates in V_4. Then $\{\{3,6\},\{2,5\}\} \subseteq V_2 \cup V_5$, which is impossible since both $\{3,6\}$ and $\{2,5\}$ are adjacent to $\{1,4\}$, while $\{1,4\} \in V_1 \cup V_6$.

Next assume that $m = 7$ and there is a $(9,2)$-partition $(V_0, V_1, \ldots, V_8)$ of $\mathrm{KG}_2(7,2)$. Because $|V(\mathrm{KG}_2(7,2))| = 14$, there are exactly five color classes of size 2 and four color classes of size 1. Hence there exists at least one pair of consecutive color classes V_i and V_{i+1} of size 2. The intersection of all vertices in V_i and V_{i+1} is a certain integer p, $1 \leq p \leq 7$. By $(*)$, we may suppose that $V_1 = \{\{1,3\},\{1,5\}\}$ and $V_2 = \{\{1,4\},\{1,6\}\}$. Then they force $V_0 = \{\{3,5\}\}$ and $V_3 = \{\{4,6\}\}$.

If both V_4 and V_8 are singletons, then we would have three consecutive color classes of size 2, which is not allowed. Thus one of V_4 or V_8 is of size 2.

Next we claim that, besides V_1 and V_2, it is impossible to have another pair of consecutive color classes of size 2. If $|V_4| = |V_5| = 2$, then the only possibility is $V_4 = \{\{2,4\},\{2,6\}\}$ and $V_5 = \{\{2,5\},\{2,7\}\}$. They in turn force $V_6 = \{5,7\}$. Moreover, $\{4,7\}$ can only belong to V_7. Since $\{3,6\}$ is adjacent to both $\{4,7\}$ and $\{5,7\}$, we see that nowhere can $\{3,6\}$ be placed properly. By similar reasons, it is impossible to have $|V_8| = |V_7| = 2$.

If $|V_6| = |V_7| = 2$, then V_6 and V_7 are either the pairs $\{\{2,4\},\{2,6\}\}$ and $\{\{2,5\},\{2,7\}\}$ or the pairs $\{\{2,7\},\{4,7\}\}$ and $\{\{3,7\},\{5,7\}\}$. In any case, $\{3,5\}$ or $\{4,6\}$ would be forced to occupy an adjacent color class. However, this is not allowed since they have already appeared in V_0 and V_2. A similar argument can be used to show that it is impossible to have $|V_5| = |V_6| = 2$.

Therefore, the only case remaining to be considered is when $|V_4| = |V_6| = |V_8| = 2$ and $|V_5| = |V_7| = 1$. In this case, the vertices of V_4 must belong to $\{\{2,4\},\{2,6\},\{3,6\},\{4,7\}\}$. Since $\{3,6\}$ cannot be placed in V_4 with $\{2,6\}$ by $(*)$ and $\{3,6\}$ is adjacent to $\{2,4\}$ and $\{4,7\}$, it follows that $V_4 = \{\{2,4\},\{2,6\}\}$ or $V_4 = \{\{2,4\},\{4,7\}\}$, which in turn forces $V_5 = \{\{2,5\}\}$ or $V_5 = \{\{2,7\}\}$. By similar reasons, we have $V_8 = \{\{2,5\},\{5,7\}\}$ or $V_8 = \{\{3,5\},\{5,7\}\}$, which in turn forces $V_7 = \{\{2,7\}\}$ or $V_7 = \{\{4,7\}\}$. We see that nowhere can $\{3,6\}$ be placed.

Case 2. For all i, $|V_i| = 1$.

By a similar argument used at the beginning of Case 2 in the proof of Theorem 6, we have $d \leq 3$.

Note that $k = |V(\mathrm{KG}_2(m,2))|$ in this case. If $d = 2$, then $2m - 5 = k = m(m-3)/2$, i.e., $m^2 - 7m + 10 = 0$. No integer $m \geq 6$ satisfies the last identity. If $d = 3$, then we have two possibilities: (i) $k/3 = m - 2 - \frac{1}{3}$ or (ii) $k/3 = m - 2 - \frac{2}{3}$. For (i), the derived identity is $m^2 - 9m + 14 = 0$ and $m = 7$ is the only possible solution. For (ii), the derived identity $m^2 - 9m + 16 = 0$ has no integer solutions.

Assume that $m = 7$ and there is a $(14,3)$-partition $(V_0, V_1, \ldots, V_{13})$ of $\mathrm{KG}_2(7,2)$. Suppose that $V_0 = \{\{x,y\}\}$ and $V_{13} = \{\{x,z\}\}$. Since the unique vertex of V_1 is adjacent to the unique vertex of V_{12}, we may suppose that $V_{12} = \{\{y,z\}\}$ and $V_1 = \{\{x,w\}\}$ for distinct numbers x, y, z, and w. This forces $V_2 = \{\{y,w\}\}$.

Note that, for each number $x \in [7]$, there are exactly four vertices in $\mathrm{KG}_2(7,2)$ that contain x: $\{x, x \pm 2\}$, $\{x, x \pm 3\}$ (additions modulo 7). If $y = x \pm 3$, then both z and w would be forced to equal $x \mp 2$, a contradiction. It follows that $y = x \pm 2$.

Indeed, the above argument also shows that, if there are three consecutive color classes occupied by three vertices with a number s in common, and if $\{s, t\}$ belongs to the middle of these three classes, then $s = t \pm 2$.

Because $V_0 = \{x, y\}$, we see that V_{11}, V_{12}, and V_{13} are occupied by vertices with z as a common number, and $\{y, z\}$ belongs to the middle of the three classes; and V_1, V_2, and V_3 are occupied by vertices with w as a common number, and $\{y, w\}$ belongs to the middle of the three classes. Therefore, by the previous paragraph, we have $z = y \pm 2$ and $w = y \pm 2$. However, in view of $y = x \pm 2$, z must equal w since x is different from z and w. Then we have arrived at a contradiction because z and w were chosen to be distinct. $\square$

We conclude this paper by proposing the following.

Problem 1 *Given positive integer $n > 1$, does there exist a number $t(n)$ such that $\chi_c(\mathrm{KG}_2(m,n)) = \chi(\mathrm{KG}_2(m,n))$ holds for all $m \geq t(n)$?*

Problem 2 *If the answer to* Problem 1 *is positive, then what is the smallest value for $t(n)$?*

Note that $t(n) > 2n+1$ if it exists, and $t(n)$ is undetermined except we have shown $t(2) = 6$.

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