Circular Chromatic Number for Iterated Mycielski Graphs *

Daphne Der-Fen Liu
Department of Mathematics
California State University, Los Angeles
Los Angeles, CA 90032, USA
Emil: dliu@calstatela.edu

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Abstract

For a graph G, let $\mathrm{M}(G)$ denote the Mycielski graph of G. The t-th iterated Mycielski graph of G, $\mathrm{M}^t(G)$, is defined recursively by $\mathrm{M}^0(G) = G$, and $\mathrm{M}^t(G)= \mathrm{M}(\mathrm{M}^{t-1}(G))$ for $t \geq 1$. Let $\chi_c(G)$ denote the circular chromatic number of G. We prove two main results: 1) If G has a universal vertex x, then $\chi_c(\mathrm{M}(G)) = \chi(\mathrm{M}(G))$ if $\chi_c(G-x) > \chi(G)-1/2$ and G is not a star, otherwise $\chi_c(\mathrm{M}(G)) = \chi(\mathrm{M}(G)) - 1/2$; and 2) $\chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m))$ if $m \geq 2^{t-1} + 2t - 2$ and $t \geq 2$.

Keywords: Mycielski graphs, circular chromatic number, chromatic number.

1 Introduction

For a positive integer k and two real numbers, $0 \leq x,\ y < k$, the *circular difference modular* k between x and y is defined by

$$|x-y|_k = \min\ \{|x-y|,\ k-|x-y|\}.$$

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Let $G = (V, E)$ be a graph. For two positive integers, $k \geq 2d$, a (k,d)*-coloring* of G is a function $f : V(G) \to \{0, 1, 2, 3, \cdots, k-1\}$ such that if two vertices u and v are adjacent, then

$$|f(u) - f(v)|_k \geq d.$$

The *circular chromatic number* is defined as

$$\chi_c(G) = \inf \{k/d : \text{ there exists a } (k,d)\text{-coloring of } G\}.$$

It is well-known that for any graph with at least one edge, the infimum in the definition above can be replaced by minimum (cf. [9]). The circular chromatic number was first introduced by Vince [7] under the name *star chromatic number*, and has been studied intensively since then; Zhu [9] provides a comprehensive survey.

Let $\chi(G)$ denote the chromatic number of G. It is true that for any G, we have (cf. [9]):

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G), \text{ and } \chi(G) = \lceil \chi_c(G) \rceil.$$

A natural question to ask is for what graphs G does the equality $\chi_c(G) = \chi(G)$ hold? This has been a major research area in the study of circular coloring. In this note, we shall examine this equality for the Mycielski graphs.

The classic Mycielski's theorem asserts the existence of triangle-free graphs with chromatic numbers as large as possible [6]. For a graph G, let $[V(G)]'$ be a copy of $V(G)$ (i.e. $[V(G)]' = \{v' : v \in V(G)\}$) and let u be a new vertex. The *Mycielski graph* of G, denoted by $\mathrm{M}(G)$, has as the vertex set $V(G) \cup [V(G)]' \cup \{u\}$, and the edge set

$$E(G) \cup \{xy' : xy \in E(G)\} \cup \{y'u : y \in V(G)\}.$$

In $\mathrm{M}(G)$, the new vertex u is called the *root*, and for each $y \in V(G)$, the copy of y, y', is called the *twin* of y, and vice versa. The iterated Mycielski graph is defined by $\mathrm{M}^t(G) = \mathrm{M}(\mathrm{M}^{t-1}(G))$. It is well-known that for any G, $\chi(\mathrm{M}(G)) = \chi(G) + 1$ and $\omega(\mathrm{M}(G)) = \omega(G)$, where $\omega(G)$ is the clique size of G. Hence, for any $t \geq 1$, $\mathrm{M}^t(K_2)$ is a triangle-free graph with chromatic number $t + 2$.

Comparing to the simple formula of the chromatic number for Mycielski graphs, the circular chromatic number for Mycielski graphs seems more complicated. This problem has been investigated by Chang, Huang and Zhu [1], Fan [2], and Hajiabolhassan and Zhu [4]. One of the main focuses in the previous work is about the circular chromatic number for the iterated Mycielski graphs of complete graphs.

A vertex adjacent to every other vertex in G is called a *universal vertex*. In the next section, we show that for any graph G with a universal vertex x, $\chi_c(\mathrm{M}(G))$ can be solely determined by the value of $\chi_c(G-x)$, where $G-x$ is the subgraph induced by the vertex set $V(G)-\{x\}$. In Section 3, we prove that $\chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m))$ for $m \geq 2^{t-1}+2t-2$ and $t \geq 2$.

2 Graphs with a universal vertex

For a (k,d)-coloring of G, one can view the colors used, $0,1,2,\cdots,k-1$, in a *cyclic order*. For $a,b \in \{0,1,\cdots,k-1\}$, let $[a,b]_k$ denote the interval from a to b in this cyclic order. For example, if $k \geq 6$, then $[2,5]_k = \{2,3,4,5\}$ and $[5,2]_k = \{5,6,\cdots,k-1,0,1,2\}$. Throughout the note, all the calculations are carried out modulo k. For instance, $[4,8]_6 = [4,2]_6$.

It is known that if $\chi_c(G) = k/d$ and $\gcd(k,d)=1$, then any (k,d)-coloring f for G is onto (cf. [3, 9]). Thus, f can be viewed as a partition of $V(G)$ into non-empty subsets V_i where $V_i = \{v : f(v) = i\}$, $i = 0,1,2,\cdots,k-1$. Using this concept, Fan [2] obtained the following result.

Lemma 1 *If $\chi_c(G) = k/d$, gcd$(k,d)=1$, then $\deg_G(v) \leq |V(G)|-2d+1$ for any $v \in V(G)$.*

For a graph G, let f be a (k,d)-coloring for the Mycielski graph of G. Without loss of generality, we may assume that $f(u)=0$ (u is the root). Let A denote the interval $[k-d+1,d-1]_k$ and $B = \{0,1,2,\cdots,k-1\}-A$. Then $f(v') \in B$ for any $v \in V(G)$. Suppose $f(v) \in B$ and $f(v') \neq f(v)$ for some $v \in V(G)$. Then one can get a (k,d)-coloring f' from f by only changing the color for v', $f'(v') = f(v)$; while others are kept the same. This process can be repeated and the following lemma, observed in [2], is true.

Lemma 2 *In $\mathrm{M}(G)$, let u be the root, and for any $v \in V(G)$, let v' be the twin of v. Suppose $\chi_c(\mathrm{M}(G)) = k/d$, gcd$(k,d)=1$, and $d \geq 2$. Then there exists a (k,d)-coloring f of $\mathrm{M}(G)$ such that $f(u)=0$, and $f(v)=f(v')$ if $f(v) \notin [k-d+1,d-1]_k$.*

In general, the value of $\chi_c(\mathrm{M}(G))$ can not be determined by $\chi_c(G)$, however, the next result indicates that for graphs with at least one universal vertex x, the value of $\chi_c(\mathrm{M}(G))$ can be solely determined by $\chi_c(G-x)$.

Theorem 3 *Let x be a universal vertex in G. Then*

$$\chi_c(\mathrm{M}(G)) = \begin{cases} \chi(\mathrm{M}(G)) - 1/2, & \text{if } \chi_c(G-x) \le \chi(G-x) - 1/2, \text{ or } G \text{ is a star;} \\ \chi(\mathrm{M}(G)), & \text{otherwise.} \end{cases}$$

Proof. Let $\chi(G-x) = m$. Then $\chi(G) = m+1$ and $\chi(\mathrm{M}(G)) = m+2$. For any $v \in V(G)$, we denote the twin of v in $\mathrm{M}(G)$ by v'.

If $m = 1$, then G is a star, x is the center, and $\chi_c(\mathrm{M}(G)) \ge 5/2$ as $\mathrm{M}(G)$ contains an odd cycle. Moreover, there exists a $(5,2)$-coloring for $\mathrm{M}(G)$, defined as: $f(x) = 1$; $f(x') = 2$; $f(u) = 0$; and for any $v \in V(G) - \{x\}$, let $f(v) = 4$ and $f(v') = 3$. Hence, $\chi_c(\mathrm{M}(G)) = 5/2 = \chi(\mathrm{M}(G)) - 1/2$.

Assume $m \ge 2$. Because $\deg_{\mathrm{M}(G)}(x) = |V(\mathrm{M}(G))| - 3$, by Lemma 1, one gets

$$\chi_c(\mathrm{M}(G)) \in \{\chi(\mathrm{M}(G)) - 1/2,\ \chi(\mathrm{M}(G))\}.$$

Hence, it suffices to verify the following two claims.

Claim 1. If $\chi_c(\mathrm{M}(G)) = \chi(\mathrm{M}(G)) - 1/2 = (2m+3)/2$, then $\chi_c(G-x) \le (2m-1)/2$.

Let f be a $(2m+3,2)$-coloring for $\mathrm{M}(G)$, satisfying Lemma 2. Set

$$V_i = \{v : f(v) = i\},\ i = 0, 1, 2, \cdots, 2m+2.$$

It suffices to find a $(2m-1,2)$-coloring for $G-x$. In $\mathrm{M}(G)$, the only vertices not adjacent to x are x' and u. Hence, if $x \in V_i$, then $V_{i-1} \cup V_{i+1} \subseteq \{u, x'\}$. As each V_j is non-empty, by symmetry, we may assume that $V_{i-1} = \{u\}$ and $V_{i+1} = \{x'\}$. Because $u \in V_0$, so $i = 1$, implying $V_0 = \{u\}$, $V_1 = \{x\}$ and $V_2 = \{x'\}$. Since x' is adjacent to all vertices in $V(G) - \{x\}$, we conclude that $V_3 \subseteq \{v' : f(v) = 2m+2,\ v \in V(G)\}$.

Now, define a mapping $f' : V(G-x) \to \{0, 1, 2, \cdots 2m-2\}$ by:

$$f'(v) = \begin{cases} f(v') - 3, & \text{if } f(v) = 2m+2; \\ f(v) - 3, & \text{otherwise.} \end{cases}$$

By the definition of Mycielski graphs and the assumption that f is a $(2m+3,2)$-coloring for $\mathrm{M}(G)$, satisfying Lemma 2, it can be easily verified that f' is a $(2m-1,2)$-coloring for $G-x$.

Claim 2. If $\chi_c(G - x) \le (2m - 1)/2$, then $\chi_c(\mathrm{M}(G)) = (2m + 3)/2$.

It is known that for any graph G, $\chi_c(G) \le k/d$ if and only if G admits a (k, d)-coloring [8]. Hence, we assume that there exists a $(2m - 1, 2)$-coloring f for $G - x$.

Define a mapping $f' : V(\mathrm{M}(G)) \to \{0, 1, 2, \cdots, 2m+2\}$ by: $f'(u) = 0$; $f'(x) = 1$; $f'(x') = 2$; and for any $v \in V(G) - \{x\}$,

$$\begin{cases} f'(v) = 2m + 2, f'(v') = 3, & \text{if } f(v) = 2m - 2, \\ f'(v) = f'(v') = f(v) + 4, & \text{if } f(v) \neq 2m - 2. \end{cases}$$

By definition, it can be verified that f' is a $(2m + 3, 2)$-coloring for $\mathrm{M}(G)$, as $m \ge 2$. We leave the details to the reader. ∎

Theorem 3 implies that $\chi(\mathrm{M}(K_m)) = m+1$ for all $m \ge 3$ (see [1]). Furthermore, it also gives the value of the circular chromatic number for odd wheels. The *n-wheel* W_n is the join of a cycle C_n and a universal vertex. For odd wheels, Lih et al. [5] proved that $\chi_c(\mathrm{M}(W_{2n+1})) \le 9/2$, and conjectured that the equality holds for all $n \ge 2$. By the fact that $\chi_c(C_{2n+1}) = 5/2$, the conjecture can be confirmed directly from Theorem 3. Note that the circular chromatic number for odd wheels is claimed to be 4.5 in [4], but no proof was given there.

Corollary 4 *For any $n \ge 2$, $\chi_c(\mathrm{M}(W_{2n+1})) = 9/2$.*

Fan [2] proved that if G contains three universal vertices, then $\chi_c(\mathrm{M}(G)) = \chi(\mathrm{M}(G))$. Hajiabolhassan and Zhu [4] strengthened this result by weakening the hypothesis to two universal vertices, for graphs with at least three vertices. This can be further generalized. Let $\Delta(G)$ denote the maximum degree of a vertex in G.

Theorem 5 *Let G be a graph on n vertices, $n \ge 4$. If G has a universal vertex and a vertex of degree $n - 2$, then $\chi_c(\mathrm{M}(G)) = \chi(\mathrm{M}(G))$.*

Proof. Let x be a universal vertex of G. Because G has a vertex of degree $n - 2$, we have $\Delta(G - x) \ge |V(G - x)| - 2$. By Lemma 1, $\chi_c(G - x) = \chi(G - x)$. Hence, the result follows from Theorem 3. ∎

3 Iterated Mycielski Cliques

Hajiabolhassan and Zhu [4] proved that $\chi_c(\mathrm{M}^t(G)) = \chi(\mathrm{M}^t(G))$ if G has at least 2^t+2 universal vertices. This implies:

$$\text{If } m \geq 2^t + 2, \text{ then } \chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m)).$$

We show that the result in the above can be strengthened to the following:

Theorem 6 *If $m \geq 2^{t-1} + 2t - 2$ and $t \geq 2$, then $\chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m))$.*

To prove Theorem 6, we make use of the following three lemmas. The first one was proved in [4], the second follows from a result (Theorem 11) in [4], and the third is from the definition of Mycielski graphs.

Lemma 7 *[4] Let $G = (V, E)$ be a graph with $\chi_c(G) = k/d$, $gcd(k, d) = 1$. Suppose f is a (k, d)-coloring for G. Then for each $i \in \{0, 1, 2, \cdots, k-1\}$, there exists an edge $xy \in E(G)$ such that $f(x) = i$ and $f(y) = i + d$. The addition is taken modulo k.*

Lemma 8 *[4] Let G be a graph with m universal vertices, $m \geq 2$. If $\chi_c(\mathrm{M}^t(G)) = k/d$, $\gcd(k, d) = 1$ and $t \geq 1$, then*

$$(m - 3)(d - 1) \leq 2^t - 2.$$

Lemma 9 *Let X be a clique of size k in G and $x \in X$. Let x^* be a copy (i.e. the twin or the twin of the twin, etc.) of x in any level of $\mathrm{M}^t(G)$. Then $(X - \{x\}) \cup x^*$ induces a clique of size k in $\mathrm{M}^t(G)$.*

Proof of Theorem 6) Assume to the contrary that $\chi_c(\mathrm{M}^t(K_m)) = k/d < \chi(\mathrm{M}^t(K_m))$, where $\gcd(k, d) = 1$. Then $d \geq 2$. Let $V(K_m) = X = \{x_1, x_2, \cdots, x_m\}$.

We first claim that $d \neq 2$. Suppose to the contrary that $d = 2$. Then,

$$\chi_c(\mathrm{M}^t(K_m)) = k/2 = \chi(\mathrm{M}^t(K_m)) - 1/2 = (2t + 2m - 1)/2.$$

So, $k = 2t + 2m - 1$.

Let f be a $(k, 2)$-coloring for $\mathrm{M}^t(K_m)$, satisfying Lemma 2 (regarding $\mathrm{M}^t(K_m)$ as the Mycielski graph of $\mathrm{M}^{t-1}(K_m)$). Without loss of generality, we assume $f(x_1) < f(x_2) < \cdots < f(x_m)$. Let U be the set of *all* the roots (i.e., roots and their copies in various levels) in $\mathrm{M}^{t-1}(K_m)$. In $\mathrm{M}^t(K_m) = \mathrm{M}(\mathrm{M}^{t-1}(K_m))$, let U' and X' be the twins of U and X, respectively, let u^* be the new root, and $U^* = U \cup U' \cup \{u^*\}$. Then, $|U| = 2^{t-1} - 1$ and $|U^*| = 2^t - 1$. Set

$$A = \{v \in V(\mathrm{M}^{t-1}(K_m)) : f(v) \in \{k-1, 0, 1\}\} \text{ and } B = V(\mathrm{M}^{t-1}(K_m)) - A.$$

Hence, $A \cap X \subseteq \{x_1, x_m\}$, and $x_l \in B$ for any $2 \le l \le m-1$. So, by Lemma 2, $f(x'_l) = f(x_l)$ for all $2 \le l \le m-1$.

Define the following:

$$\hat{x}_i = \begin{cases} x_i, & \text{if } x_i \in B, \\ x'_i, & \text{if } x_i \in A; \end{cases}$$

$$\hat{u} = \begin{cases} u, & \text{if } u \in B, \\ u', & \text{if } u \in A; \end{cases}$$

$$\hat{X} = \{\hat{x}_i : i = 1, 2, \cdots, m\}; \text{ and } \hat{U} = \{\hat{u} : u \in U\}.$$

Then $f(\hat{X}), f(\hat{U}) \subseteq [2, k-2]_k$ and $|\hat{U}| = |U| = 2^{t-1} - 1$. Moreover, $|f(\hat{X})| = m$ except when $f(x'_1) = f(x'_m)$, when $|f(\hat{X})| = m - 1$.

Line up the vertices in $\hat{X}$ by $c_1, c_2, c_3, \cdots$ such that $f(c_1) \le f(c_2) \le f(c_3) \le \cdots$. Let $f(x'_1) = f(c_a)$ and $f(x'_m) = f(c_b)$ for some a and b. It suffices to consider the following three cases.

Case 1. $|a - b| \ge 2$, or $|a - b| = 1$ with $x_1 \in B$ or $x_m \in B$ Then $|f(\hat{X})| = m$. Define the intervals I_i, for $i = 1, 2, \cdots, m-1$, by:

$$I_i = [f(c_i) + 1, f(c_{i+1}) - 1]_k.$$

Then $|I_i| \ge 1$ for all i. By Lemma 9, the fact $f(x'_l) = f(x_l)$ for all $2 \le l \le m-1$, and the assumption for this case, we conclude that if $\hat{x}_j = c_i$ for some i and j, then the colors in the set $[f(c_i) - 1, f(c_i) + 1]_k \cap f(B)$ can only be assigned to $\hat{U}$ or copies of x_j in various levels in $\mathrm{M}^t(G)$. This implies:

$$\text{If } I_i \cap f(\hat{U}) = \emptyset, \text{ then } |I_i| \ge 2. \tag{3.1}$$

Let $M = m - 1$. Because $|\hat{U}| \leq 2^{t-1} - 1$, among these M intervals of I_i's, at least $M - (2^{t-1} - 1)$ of them have $|I_i| \geq 2$. Hence, we get

$$\begin{aligned} k = 2m + 2t - 1 &\geq 3 + |f(\hat{X})| + \sum_{i=1}^{M} |I_i| \\ &\geq 3 + |f(\hat{X})| + 2^{t-1} - 1 + 2(M - 2^{t-1} + 1) \\ &= |f(\hat{X})| - 2^{t-1} + 2M + 4 \qquad (3.2) \end{aligned}$$

As $|f(\hat{X})| = m$ and $M = m - 1$, calculation from (3.2) implies that $m \leq 2^{t-1} + 2t - 3$, a contradiction.

Case 2. $f(x'_1) = f(x'_m)$ Then $f(x_1) = 1$, $f(x_m) = k - 1$, $|f(\hat{X})| = m - 1$, and $3 \leq f(x'_1) = f(x'_m) \leq k - 3$.

By Lemma 9, $f(c_1) \geq 3$ and $f(c_m) \leq k - 3$. Since $f(x'_1) = f(x'_m) = f(c_j) = f(c_{j+1})$ for some $1 \leq j \leq m - 1$, we define m intervals I_i by: $I_1 = [2, f(c_1) - 1]_k$; $I_m = [f(c_m) + 1, k - 2]_k$; and

$$\begin{aligned} I_{i+1} &= [f(c_i) + 1, f(c_{i+1}) - 1]_k, \quad i = 1, 2, 3, j - 1; \\ I_i &= [f(c_i) + 1, f(c_{i+1}) - 1]_k, \quad i = j + 1, j + 2, \cdots, m - 1. \end{aligned}$$

Then $|I_i| \geq 1$ for all i, and similar to Case 1, both (3.1) and (3.2) hold, for $M = m$. As $|f(\hat{X})| = m - 1$, calculation from (3.2) implies that $m \leq 2^{t-1} + 2t - 4$, a contradiction.

Case 3. $|a - b| = 1$ and $\{x_1, x_m\} \subseteq A$ Then $f(x_1) = 1$, $f(x_m) = k - 1$, and $|f(\hat{X})| = m$. Assume $b = a + 1$. (The case for $a = b + 1$ is symmetric.)

Assume $2 \leq a \leq m - 2$. By Lemma 9, $f(c_1) \geq 3$ and $f(c_m) \leq k - 3$. Define m intervals by: $I_1 = [2, f(c_1) - 1]_k$; $I_m = [f(c_m) + 1, k - 2]_k$; and

$$I_{i+1} = \begin{cases} [f(c_i) + 1, f(c_{i+1}) - 1]_k, & \text{if } i = 1, 2, \cdots, a - 1; \\ [f(c_{i+1}) + 1, f(c_{i+2}) - 1]_k, & \text{if } i = a, a + 1, \cdots, m - 2. \end{cases}$$

Then $|I_i| \geq 1$ for all i, and similar to Case 1, both (3.1) and (3.2) hold, for $M = m$. As $|f(\hat{X})| = m$, calculation from (3.2) implies that $m \leq 2^{t-1} + 2t - 5$, a contradiction.

If $a = 1$, then $f(c_m) \leq k - 3$. (The case for $a = m - 1$ is symmetric.) Define $(m - 1)$ intervals I_i by: $I_{m-1} = [f(c_m) + 1, k - 2]_k$; and

$$I_i = [f(c_{i+1}) + 1, f(c_{i+2}) - 1]_k, \quad i = 1, 2, \cdots, m - 2.$$

Again, an easy calculation, similar to the above, on these $m-1$ intervals leads to $m \le 2^{t-1}+2t-3$, a contradiction.

We conclude, from the above three cases, that $d \ne 2$.

Assume $d \ge 3$. By the assumptions that $m \ge 2^{t-1}+2t-2$ and $t \ge 2$, and by Lemma 8, it must be that $m=4$, $t=2$ and $d=3$. Assume $\chi_c(\mathrm{M}^2(K_4)) = k/3$. Let f be a $(k,3)$-coloring for $\mathrm{M}^2(K_4)$. Denote the vertices of K_4 by $X=\{x_1,x_2,x_3,x_4\}$, and their corresponding twins in $\mathrm{M}(K_4)$ by $Y=\{y_1,y_2,y_3,y_4\}$. Let v be the root of $\mathrm{M}(K_4)$. That is, $V(\mathrm{M}(K_4)) = X \cup Y \cup \{v\}$. For each $w \in V(\mathrm{M}(K_4))$, let w' be the twin of w in $\mathrm{M}^2(K_4)$. We assume that f satisfies Lemma 2. Set

$$A = \{w \in V(\mathrm{M}(K_4)) : f(w) \in \{k-1, k-2, 0, 1, 2\}\} \ \text{ and } \ B = V(\mathrm{M}(K_4)) - A.$$

Hence, $A \cap X \subseteq \{x_1, x_4\}$, and $x_2, x_3 \in B$. By Lemma 2, $f(x_2') = f(x_2)$ and $f(x_3') = f(x_3)$. Note that, $f(x_3) \le k-4$, since $f(x_4) \le k-1$. If $f(x_2) \ge 4$, then we have four distinct colors $\{f(x_2)-1,\ f(x_2)+1,\ f(x_3)-1,\ f(x_3)+1\}$, while only three sets of vertices, $\{y_2, y_2'\}$ (either $f(y_2) = f(y_2')$ or $y_2 \in A$), $\{y_3, y_3'\}$ and $\{v, v'\}$, that can be assigned by these colors. This contradicts that f is onto.

So, it must be that $f(x_2) = 3$, which implies $f(x_1) = 0$. Then the colors 1 and 2 can only be possibly assigned to the vertex v, contradicting that f is onto. ∎

An immediate corollary of Theorem 6 is the confirmation of the following conjecture, for the case $t=2$.

Conjecture 1 *[1] Let t be a positive integer. If $m \ge t+2$, then $\chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m))$.*

Note that Conjecture 1 has been confirmed by Chang et al. [1] for the cases when $t=1,2$, with involved calculations. A shorter proof for the case $t=1$ was given by Fan [2]. For $t \ge 3$, the conjecture remains open. In particular, the following is of interest.

Problem 2 *For any $t \ge 1$, what is $n(t)$, the smallest positive integer n such that the equality $\chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m))$ holds for all $m \ge n$?*

It is known [1] that $n(1) = 2$ and $n(2) = 4$. For $t \geq 3$, combining a result in [1] with Theorem 6, we have

$$t + 2 \leq n(t) \leq 2^{t-1} + 2t - 2,$$

however, the exact value of $n(t)$ is still unknown and worthy of further investigation.

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