Comprehensive Examination – Topology

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Do any five of the problems that follow. Each problem is worth 20 points. The set of positive integers is denoted by $N$, the set of rationals by $Q$, and the set of real numbers by $R$. The notation $A^c$ means the complement of the set $A$ with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $f : X \to Y$ be continuous.
   (a) Let $A$ be a connected subset of $X$. Prove that $f(A)$ is a connected subset of $Y$.
   (b) Give an example showing that “connected” cannot be replaced by “closed”.

2. Let $(X, d)$ be a metric space.
   (a) Let $x_0 \in X$ and let $r > 0$. Prove that the closed ball $B[x_0, r] = \{x \in X : d(x, x_0) \leq r\}$ is a closed subset in $(X, d)$.
   (b) Prove that $X$ is regular.

3. Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $f : X \to Y$ be continuous.
   (a) Assuming that $Y$ is Hausdorff, prove that the graph of $f$, $\Gamma(f) = \{(x, y) : x \in X, y = f(x)\}$ is a closed subset of $X \times Y$ equipped with the product topology.
   (b) Prove that $\Gamma(f)$ equipped with the relative topology is homeomorphic to $X$.

4. Suppose that the topological space $(X, \tau)$ has a countable base.
   (a) Show that if $\{V_\alpha : \alpha \in \Delta\}$ is a family of open, pairwise disjoint, nonempty subsets of $X$, then $\Delta$ must be countable.
   (b) Let $A$ be an uncountable subset of $X$. Prove that some point of $X$ must be a limit point of $A$. (Hint: if not, consider $A$ as a subset of $X$.)

5. Let $(X, \tau)$ be a Hausdorff space. We say that a sequence $(x_n)_{n=1}^\infty$ is convergent to $x \in X$ iff for each neighborhood $V$ of $x$ there exists $n \in N$ such that for each $k > n$ we have $x_k \in V$. In this case we write $(x_n) \to x$.
   (a) Suppose that $(x_n) \to x$ and $(x_n) \to y$. Prove that $x = y$.
   (b) Suppose that $(x_n) \to x$. Prove that $\{x_n : n \in N\} \cup \{x\}$ is a compact subset of $X$.

6. The lower limit topology on $R$, a.k.a. the Sorgenfrey topology, is the topology $\tau_L$ having as a base all half-open intervals $[a, b)$ where $a < b$.
   (a) Is the space $(R, \tau_L)$ first countable? Explain.
   (b) Is the space $(R, \tau_L)$ connected? Explain.
   (c) Is $[0, 1]$ compact as a subspace of $(R, \tau_L)$? Explain.

7. Let $(X, \tau)$ be a topological space and $(Y, \sigma)$ be compact topological space. Suppose that $F$ is a closed subset of $X \times Y$ and $\pi_1$ is the usual projection map from $X \times Y$ to $X$. Show that if $x_0 \in X \setminus \pi_1(F)$, then there exists a neighborhood $U$ of $x_0$ such that $F \cap (U \times Y) = \emptyset$. 
