Comprehensive Examination – Topology

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Do any five of the problems that follow. Each problem is worth 20 points. The set of positive integers is denoted by \( N \), the set of rationals by \( Q \), and the set of real numbers by \( R \). The notation \( A^c \) means the complement of the set \( A \) with respect to an understood universal set. The notation \( A \setminus B \) means \( \{ a : a \in A \text{ and } a \notin B \} \).

1. Prove that each metric space \((X, d)\) is normal.

2. (a) Give an example of a connected topological space that is not locally connected.
   (b) Give an example of a topological space that is not separable. Justify your answer.

3. Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces and let \( f : X \to Y \) be continuous.
   (a) Assuming that \( Y \) is Hausdorff, prove that the graph of \( f \), \( \Gamma(f) = \{(x, y) : x \in X, y = f(x)\} \)
       is a closed subset of \( X \times Y \) equipped with the product topology.
   (b) Prove that \( \Gamma(f) \) equipped with the relative topology is homeomorphic to \( X \).

4. Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces and let \( f : X \to Y \). Prove that \( f \) is continuous if
   and only if for each subset \( A \) of \( X \) we have \( f(A^c) \subseteq f(A) \). Here \( E \) denotes the closure of \( E \).

5. Let \((X, \tau)\) be a compact Hausdorff space.
   (a) Let \( F \) be a nonempty closed subset of \( X \). Prove that \( F \) equipped with the relative topology is compact.
   (b) Let \( f : X \to X \) be continuous, one-to-one, and onto. Prove that \( f \) is a homeomorphism.

6. (a) What is meant by \( \text{diam}(A) \), the diameter of a nonempty subset \( A \) of a metric space \((X, d)\)?
   (b) Let \(< A_n >\) be a sequence of nonempty closed sets in a complete metric space \((X, d)\) such
       that for any \( n \) \( A_{n+1} \subseteq A_n \) and \( \lim_{n \to \infty} \text{diam}(A) = 0 \). Prove that \( \bigcap_{n=1}^{\infty} A_n \neq \emptyset \).

7. Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces.
   (a) Suppose that \( A \) is closed subset of \( X \) and \( B \) is closed subset of \( Y \). Prove or produce a
       counterexample: \( A \times B \) is a closed subset of \( X \times Y \) equipped with the product topology.
   (b) Suppose \( E \) is a closed subset of \( X \times Y \) equipped with the product topology. Prove or
       produce a counterexample: \( \pi_X(E) = \{ x \in X : \exists y \in Y \text{ with } (x, y) \in E \} \).