Comprehensive Examination – Topology

Fall 2002
G. Beer, P. Chabot, A. Verona*

Do five problems, including the first one. Each problem is worth 20 points. The set of positive integers is denoted by $\mathbb{N}$, the set of rationals by $\mathbb{Q}$, and the set of real numbers by $\mathbb{R}$. The notation $A^c$ means the complement of the set $A$ with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Explain carefully the following concepts:
   (a) Locally connected space.
   (b) Uniformly continuous function between two metric spaces.
   (c) Normal space.
   (d) Separable space and second countable space.
   (e) Compact topological space.
   (f) Basis for a topology.

2. Let $d_1$ and $d_2$ be metrics on the same space $X$ and let $\tau_1$ and $\tau_2$ be the corresponding topologies. Prove that the following are equivalent:
   (a) Whenever $(x_n) \xrightarrow{d_1} x$ then $(x_n) \xrightarrow{d_1} x$ (that is whenever a sequence converges with respect to $d_1$ it also converges with respect to $d_2$, and to the same limit).
   (b) $\tau_1 \subseteq \tau_2$

3. Let $X$ and $Y$ be topological spaces with $X$ connected and let $f : X \to Y$ be continuous.
   (a) Prove that $f(X)$ is connected.
   (b) Prove that $\Gamma(f) = \{(x, f(x)) : x \in X\}$ is connected as a subspace of $X \times Y$.

4. On $\mathbb{R}$ consider the following family of subsets:
   $$\tau = \{D \subseteq \mathbb{R} : D = \emptyset, \text{ or } D = \mathbb{R} \text{ or the complement of } D \text{ is countable}\}.
   $$
   (a) Prove that $\tau$ is a topology on $\mathbb{R}$.
   (b) Show that any convergent (in $\tau$) sequence is eventually constant (that is, whenever $\lim_{n \to \infty} x_n = x$ there exists $N_0$ such that $x_n = x$ if $n > N_0$).

5. Prove that a topological space is (Hausdorff) if and only if $D = \{(x, x) \in X \times X : x \in X\}$ is closed as a subspace of $X \times X$ endowed with the product topology.

6. Prove that every (Hausdorff) compact regular space is normal.