Do five problems, including the first one. Each problem is worth 20 points. The set of positive integers is denoted by $N$, the set of rationals by $Q$, and the set of real numbers by $R$. The notation $A^c$ means the complement of the set $A$ with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Explain carefully 5 of the following concepts:
   (a) Connected component of a topological space.
   (b) Uniformly continuous function between two metric spaces.
   (c) Normal space.
   (d) Product topology.
   (e) Compact topological space.
   (f) Basis for a topology.

2. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ be a continuous function. Define $\rho : X \times X \to R$ by $\rho(x_1, x_2) = d_X(x_1, x_2) + d_Y(f(x_1), f(x_2))$. Prove that
   (a) $\rho$ is a metric on $X$.
   (b) The metrics $\rho$ and $d_X$ are equivalent.

3. Let $A, B$ be subsets of a topological space $X$. Prove or disprove each of the following statements:
   (a) $\text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B)$.
   (b) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.
   (c) $\overline{A} \cap \overline{B} = \emptyset$ implies that $A \cap \overline{B} = \emptyset$.

4. Let $X$ and $Y$ be topological spaces. Suppose that $X$ is normal and $f : X \to Y$ is continuous, open, and onto. Prove that $Y$ is normal.

5. Let $X$ be a topological space and $A$ be a subset of $X$.
   (a) Define $\text{bd}(A)$, the boundary of $A$.
   (b) Use the definition in (a) to prove that: $A$ is open if and only if $A \cap \text{bd}(A) = \emptyset$.
   (c) Use the definition in (a) to prove that: $A$ is closed if and only if $\text{bd}(A) \subseteq A$.

6. Let $X, Y$ be metric spaces, $X$ being compact. Prove that a continuous mapping $f : X \to Y$ is uniformly continuous.

7. Let $f : X \to Y$ be a continuous mapping.
   (a) Prove or disprove: If $C \subseteq Y$ is connected then $f^{-1}(C)$ is connected.
   (b) Prove or disprove: If $C \subseteq X$ is connected then $f(C)$ is connected.
   (c) State a theorem from Calculus which is a particular case of (a) or (b).