Do five of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: \( \mathbb{C} \) denotes the set of complex numbers.
\( \mathbb{R} \) denotes the set of real numbers.
\( \text{Re}(z) \) denotes the real part of the complex number \( z \).
\( \text{Im}(z) \) denotes the imaginary part of the complex number \( z \).
\( \bar{z} \) denotes the complex conjugate of the complex number \( z \).
\( |z| \) denotes the absolute value of the complex number \( z \).
\( \{x_n\}_{n=1}^{\infty} \) denotes a sequence \( x_1, x_2, x_3, \ldots \).
\( (X, A) \) denotes a set \( X \) together with a \( \sigma \)-algebra of subsets of \( X \).
\( (X, A, \mu) \) denotes a set \( X \) together with a \( \sigma \)-algebra of subsets of \( X \) and a non-negative measure \( \mu \) defined on \( A \).

If \( A \) and \( B \) are sets, then \( A \setminus B \) denotes the set difference \( A \setminus B = \{ x \in A : x \notin B \} \).
Spring 2001 # 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real valued function on $\mathbb{R}$ such that $\lim_{x \to \infty} xf(x) = L$ with $L \in \mathbb{R}$. Show that $\lim_{x \to \infty} f(x) = 0$.

Spring 2001 # 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real valued function on $\mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for all $x$ and $y$ in $\mathbb{R}$. Assume that $\lim_{x \to 0} f(x) = L$ exists in $\mathbb{R}$.
   a. Show that $L = 0$
       (Suggestion: Consider $f(1) = f\left(n \cdot \frac{1}{n}\right)$)
   b. Show $f$ is continuous at every point of $\mathbb{R}$.

Spring 2001 # 3. a. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $|f'(t)| \leq M < \infty$ for all $t$ in $\mathbb{R}$. Show that $f$ is uniformly continuous on $\mathbb{R}$. (You may use basic theorems from calculus.)
   b. Give an example of a uniformly continuous function on $[0, 1]$ which is differentiable on the open interval $(0, 1)$ but whose derivative is not bounded on $(0, 1)$. Justify your answer.

Spring 2001 # 4. Let $(\Omega, \Sigma)$ be a measurable space. (You may assume that $\Omega$ is the real line with Lebesgue measure if you wish.)
   a. Define what it means for a function $f : \Omega \to \mathbb{R}$ to be measurable.
   b. Use your definition to show that if $f : \Omega \to \mathbb{R}$ is a measurable function, then so is the function $f^2 : \Omega \to \mathbb{R}$ defined by $f^2(t) = (f(t))^2$ for all $t$ in $\Omega$.
   c. Is the converse true? If $f^2$ is measurable must $f$ be measurable?

Spring 2001 # 5. If $f$ and $g$ are (Lebesgue) measurable real-valued functions and $(f^2 + g^2)^{1/2}$ is integrable, prove that $f + g$ is (Lebesgue) integrable.

Spring 2001 # 6. For each $n = 1, 2, 3, \ldots$, define a function $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{1 + x^{2n}}.$$

   a. Explain how you know that $f_n$ is integrable on $[0, \infty)$.
   b. Show that $\lim_{n \to \infty} \int_0^\infty \frac{1}{1 + x^{2n}} \, dx = 1$.

Spring 2001 # 7. Suppose $h$ is an integrable function on a measurable domain $E$ such that $h(t) \geq 0$ for almost every $t$ in $E$. Show that if $0 < \beta < \infty$, then

$$\mu(\{t \in E \mid h(t) \geq \beta\}) \leq \frac{1}{\beta} \int_E h(t) \, d\mu(t).$$

(You may assume that $\mu$ is Lebesgue measure on the real line if you wish.)
Spring 2001 # 8. Suppose $A_1, A_2, A_3, \ldots$ are measurable sets in a finite measure space $(\Omega, \Sigma, \mu)$ such that $\sum_{k=1}^{\infty} \mu(A_k \setminus A_{k-1}) < \infty$ and $\lim_{n \to \infty} \mu(A_n) = 0$.

Prove that $\mu(\limsup A_k) = 0$

Note: (1) You may consider $\Omega$ to be a bounded interval in the real line with Lebesgue measure if you wish.

(2) $\limsup A_k = \{ t \in \Omega : t \in A_k \text{ for infinitely many indices } k \} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

End of Exam