PART 1 (Do two problems)

I-1 a. Let \( A = \begin{bmatrix} 1 & k & k \\ k & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \), where \( k > 0 \).

(i) Show that the spectral radius of the Jacobi iteration matrix for \( A \) is \( k\sqrt{2} \). [6%]

(ii) For what values of \( k \) does Jacobi iteration converge? What is the rate of convergence? [3%]

(iii) For \( k = 0.5 \), how many iterations are needed for Jacobi iteration (used to approximate the solution to \( Ax = b \)) to reach an accuracy of \( 10^{-5} \)? [5%]

b. For a general linear system \( Ax = b \), where \( A \) is \( n \times n \), and a general splitting \( A = M - N \), show that the iterative method \( Mx^{(k+1)} = Nx^{(k)} + b \) converges if the spectral radius of the matrix \( M^{-1}N \) is less than 1. [8%]

c. If the iteration matrix in part \( b \) is strictly diagonally dominant, must the corresponding iterative method converge? Explain your answer. [3%]
I-2  

a. Let \( A = \begin{bmatrix} 6 & 4 & 2 \\ 3 & -2 & -1 \\ 3 & 4 & 1 \end{bmatrix} \).

Find the LDU decomposition of \( A \), \( A = LDU \), where \( L \) is unit lower-triangular, \( D \) is diagonal, and \( U \) is upper-triangular. [8%]

b. Let \( B \) be an \( n \times n \) matrix. Show that if \( B \) can be factored as \( B = LU \), where \( L \) is unit lower-triangular and \( U \) is upper-triangular, then this factorization is unique. [8%]

c. Given the linear system \( Cx = b \), where \( A \) has been factored as \( C = LU \), and

\[
L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix},
\]

solve \( Cx = b \) by forward/backward substitution (\( \text{without multiplying} \ L \text{ and } U \)). [6%]

d. Briefly explain the role of “partial pivoting” in the Gaussian elimination procedure: What is it and what is its purpose? [3%]

I-3  

The “Power Method” and the “QR Method” are techniques for finding approximations to the eigenvalues of a square matrix \( A \).

a. Let \( A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \) and \( x^{(0)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \). Then, \( A \) has eigenvalues 1 and 3 and corresponding eigenvectors \( [1 \ 0]^T \) and \( [1 \ 1]^T \), respectively.

(i) Apply two iterations of the Power Method to the matrix \( A \) with initial vector \( x^{(0)} \) to obtain \( x^{(2)} \), an approximation to the eigenvector of \( A \) corresponding to its dominant eigenvalue. [6%]

(ii) Will the Power Method converge to the dominant eigenvalue of \( A \) in this case? Explain why or why not. [4%]

b. (i) Obtain the QR factorization of the matrix \( A \) of part a, and use this factorization to obtain the first iteration in the QR method. [9%]

(ii) Give one advantage and one disadvantage of the Power Method compared to the QR Method for finding the eigenvalues of an \( n \times n \) matrix \( B \). [6%]
PART II (Do two problems)

II-1 Consider the following elliptic boundary-value problem in the region
D = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}:
\[
\begin{align*}
   u_{xx} + u_{yy} &= 0 \quad \text{(PDE)} \quad 0 < x < 1, 0 < y < 1 \\
   u(0, y) &= -y^2, \quad u(1, y) = 1 - y^2, \quad 0 \leq y \leq 1 \\
   u(x, 0) &= x^2, \quad u(x, 1) = x^2 - 1, \quad 0 \leq x \leq 1
\end{align*}
\]

a. Show that \( u(x, y) = x^2 - y^2 \) is an exact solution of this boundary-value problem. [5%]

b. What are the maximum and minimum values achieved by the solution, \( u \), to the given
   boundary-value problem in the region \( D \)? At what points \( (x, y) \) do they occur? [4%]

c. With \( \Delta x = \Delta y = 1/3 \), use the usual five-point difference scheme for approximating
   the given PDE to obtain a system of linear equations for solving this problem. Express
   this system in the form \( Au = b \), where \( A \) is a \( 4 \times 4 \) matrix. [12%]

d. Explain why the solution to your difference approximation in part c is unique. [4%]

II-2 a. Write a consistent explicit difference scheme for approximating the partial differential
   equation
\[
\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} + (x^2 + 1) \frac{\partial u}{\partial x}
\]
Solve your scheme for \( U_{i,j+1} \) \([= U(i\Delta x, (j+1)\Delta t)]\), using \( r = k/h^2 \), but do not show
that it is consistent. [5%]

b. Show that the simple (classical) explicit scheme for approximating \( u_t = u_{xx} \) is
   consistent. [8%]

c. Use the Fourier (von Neumann) method to show that the following scheme for
   approximating \( u_t = u_{xx} \) is unstable for all values of \( r = k/h^2 \):
\[
\frac{U_{i,j+1} - U_{i,j-1}}{2k} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}
\] [12%]
II-3 Consider the partial differential equation (PDE) \( u_{xx} + xu_{xy} - 2x^2u_{yy} = 0 \) with initial data given on the line \( y = 0 \).

a. Give the condition for determining whether a PDE of the form
   \[
   A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = 0
   \]
   is elliptic, parabolic, or hyperbolic at the point \((x,y)\), and use this condition to determine all values of \( x \) for which the given PDE is hyperbolic. \([6\%]\)

b. Determine the two characteristic directions (slopes of the characteristic curves, \(dy/dx\)) for the given PDE. \([5\%]\)

c. Find the exact values of the coordinates of the point of intersection, \( R \), of the characteristic curves through the points \( P(1,0) \) and \( Q(2,0) \). \([8\%]\)

d. Write a scheme with \( h = k \ (\Delta x = \Delta y) \) that is consistent with the given PDE. (Simplify your scheme, but do not show that it is consistent.) \([6\%]\)