Instructions: Do any 2 problems from Part I AND any 2 problems from Part II

PART I (Do two problems)

I-1 a. In solving the linear system \( Ax = b \), where \( b \) is an arbitrary 3-vector and
\[
A = \begin{bmatrix}
1 & 2 & -2 \\
1 & 1 & 1 \\
2 & 2 & 1
\end{bmatrix},
\]
determine whether or not the Jacobi iteration method converges. [9%]

b. Given the linear system \( Bx = b \), where \( B \) is a strictly diagonally dominant \( n \times n \)
matrix (and \( b \) is an arbitrary \( n \)-vector), prove that the Jacobi iteration matrix for this
system, \( B_j \), satisfies \( \| B_j \|_\infty \leq 1 \). [8%]

c. Suppose that \( C \) is an \( n \times n \) matrix that satisfies \( \| C \| < 1 \) and that \( x \) is an \( n \)-vector that
satisfies \( x = Cx + c \), where \( c \) is an arbitrary \( n \)-vector. Prove that the sequence defined by
\[
x^{(k)} = Cx^{(k-1)} + c \quad (k = 1, 2, 3, \ldots; x^{(0)} \text{ arbitrary})
\]
converges to \( x \) (that is, \( \| x^{(k)} - x \| \to 0 \) as \( k \to \infty \)). [8%]
I-2  a. Consider the following nonsingular matrix:
\[
A = \begin{bmatrix}
1 & 2 & 6 \\
4 & 8 & -1 \\
-2 & 3 & 5
\end{bmatrix}.
\]

(i) Show that A cannot be directly factored as \( A = LU \) (that is, show that this equation has no solution), where L is unit lower-triangular and U is upper-triangular. [9%]

(ii) Apply Gaussian elimination with row interchanges on A to obtain a permutation matrix P, a unit lower-triangular matrix L, and an upper-triangular matrix U such that \( PA = LU \). [9%]

b. Partial-pivoting is a technique that is commonly used in conjunction with the Gaussian elimination method when solving a system of linear equations.

(i) Briefly explain what is meant by “partial-pivoting.” [4%]

(ii) Briefly explain the purpose of using partial-pivoting in solving large linear systems by Gaussian elimination. [3%]

I-3  a. The “Power Method” and the “QR Method” are techniques for finding approximations to the eigenvalues of a square matrix A.

(i) Give sufficient conditions for the convergence of the Power Method. [4%]

(ii) Apply two iterations of the Power Method on the matrix
\[
A = \begin{bmatrix}
0 & 3 \\
1 & 2
\end{bmatrix}
\]
with initial vector \( x^{(0)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \)
to obtain \( x^{(2)} \), an approximation to the eigenvector of A corresponding to its dominant eigenvalue. [6%]

(iii) Give one advantage of the Power Method over the QR Method. [3%]

b. Find the 3 × 3 matrix B that has eigenvalues \( \lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2 \) and corresponding orthogonal eigenvectors:
\[
\begin{align*}
x_1 &= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.
\end{align*}
\]
PART II (Do two problems)

II-1 Suppose that the function \( u(x, y) \) satisfies
\[
x u_{xx} + y u_{yy} = 3
\]
inside the square region shown at the right (with \( h = \Delta x = \Delta y = 1/4 \)), and that it satisfies the indicated boundary conditions.

a. Show that the given partial differential equation is elliptic in this region. [4%]

b. Denoting the finite difference solution at grid point \( k \) by \( u_k \) (\( k = 1, 2, ..., 12 \)), write finite difference equations consistent with the given PDE at grid points 4 and 5. (You do not have to show consistency.) [8%]

c. For a function \( v(x, y) \), find constants \( c_0, c_1, \) and \( c_2 \) such that
\[
\left| \frac{\partial v}{\partial x}(x_0, y_0) - [c_0 v(x_0, y_0) + c_1 v(x_0 - h, y_0) + c_2 v(x_0 - 2h, y_0)] \right| < K h^2
\]
where \( K \) is a constant independent of \( h \). (You may assume that \( v(x, y) \) has as many partial derivatives as you need.) [8%]

d. Write an \( O(h^2) \) finite difference equation to replace the boundary condition \( u_x(1, \frac{1}{2}) = 3y \) at grid point 11. [5%]
II-2 Consider the initial-boundary value problem:

\[
\begin{align*}
\frac{u_t}{u_{xx}} & \quad 0 \leq x \leq 1, \ t > 0 \\
u(x,0) &= x & 0 \leq x \leq 1 \\
u(0,t) &= 0, \ u(1,t) = 1 & t > 0
\end{align*}
\]

Suppose we approximate the PDE \( u_t = u_{xx} \) by the finite difference scheme

\[
-\theta U_{i-1,j+1} + (1 + 2\lambda\theta)U_{i,j+1} - \theta U_{i+1,j+1} = \lambda(1 - \theta)U_{i-1,j} + (1 - 2\lambda(1 - \theta))U_{i,j} + \lambda(1 - \theta)U_{i+1,j}
\]

where \( U_{i,j} = U(ih, jk) \), \( \lambda = k/h^2 \), and \( \theta \) is a parameter with \( 0 < \theta < 1 \).

a. For which value(s) of \( \theta \) is this scheme *implicit*? [3%]

b. Describe (in one sentence each) what it means when we say that:

i. The given scheme is *consistent* with the given PDE.

ii. The given scheme is a *stable* approximation to the given PDE.

c. Let \( \theta = 0 \). For what values of \( \lambda \) is the resulting scheme convergent? (No proof is necessary.) [3%]

d. Let \( \theta = 1 \). For what values of \( \lambda \) is the resulting scheme convergent? (No proof is necessary.) [3%]

e. Taking \( h = 1/3 \), \( k = 1/6 \) (so \( \lambda = 3/2 \)), and \( \theta = 2/3 \), give the system of two algebraic equations that relates \( U_{1,j+1} \) and \( U_{2,j+1} \) to \( U_{1,j} \) and \( U_{2,j} \). [10%]

II-3 a. Given the hyperbolic partial differential equation

\( u_{tt} = u_{xx} \) \( (-\infty < x < \infty, \ t > 0) \),

show that the change of variables to the characteristic coordinates \( \xi = x + t, \ \eta = x - t \) transforms this PDE into the form

\( u_{\xi\eta}(\xi, \eta) = 0 \) \( (-\infty < \eta < \xi < \infty) \). [10%]

b. Suppose we use the usual explicit scheme with \( r = k/h = 1 \),

\[
U_{i,j+1} = U_{i+1,j} + U_{i-1,j} - U_{i,j-1}
\]

together with forward differences to approximate the initial-value problem

\[
\begin{align*}
u_{tt} &= u_{xx} & -\infty < x < \infty, \ t > 0 \\
u(x,0) &= f(x) & -\infty < x < \infty \\
u(x,0) &= g(x) & -\infty < x < \infty
\end{align*}
\]

Show that the numerical solution converges to the solution of the initial-value problem as \( h \to 0 \). [15%]