Part I: (Do two problems)

1. Let \( B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \).
   (a) By computing the spectral radius of the iteration matrix, determine whether or not
       Jacobi iteration converges in solving \( Bx = c \) for an arbitrary 3-vector \( c \).
   (b) Without doing any further work can you determine whether the Gauss-Seidel iteration
       will converge for solving \( Bx = c \) or not? Explain.
   (c) Note that the above matrix \( B \) is positive definite and tridiagonal. Based on this and
       your results for part (a) determine the spectral radius of the Gauss-Seidel iteration matrix.
   (d) Given the linear system \( Ax = b \) where \( A \) is an \( n \times n \) matrix and \( b \) an \( n \)-vector, write
       \( A = M - N \) where \( M \) is nonsingular and consider the iterative scheme
       \[ x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b \quad (k = 1, 2, 3, \ldots). \]
       Show that
       \[ ||x^{(k)} - x|| \leq ||G||^k||x^{(0)} - x||, \]
       where \( G = M^{-1}N \), \( x^{(0)} \) is the initial approximation, and \( x \) is the actual solution.

2. \( A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 5 & 7 & 2 \\ 2 & 7 & 14 & 3 \\ 1 & 2 & 3 & 3 \end{bmatrix} \).
   (a) Find a decomposition of \( A \) in the form \( A = R^TR \), where \( R \) is a upper triangular matrix.
   (b) For a nonsingular matrix \( M \) show that \( B = M^TM \) is positive definite.
   (c) For a positive definite matrix \( C = [c_{ij}] \) show that \( c_{ii} > 0 \).
   (d) Show that the \( C \) in part (c) is nonsingular.
3. (a) Given a $n \times n$ matrix $A$ and with $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$, where $\lambda_i$ is an eigenvalue of $A$,

(i) Describe the power method to find $\lambda_1$, and its corresponding eigenvector.

(ii) Show the convergence is linear.

(b) Describe briefly how the Rayleigh Quotient Iteration method improves the rate of convergence for the above $A$.

(c) Do two steps of the Rayleigh Quotient Iteration method on the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ and compare the theoretical and observed rates of convergence.
Part II: (Do two problems)

1. (a) Consider the first-order PDE
\[ au_x + bu_y = c, \]
where \(a, b,\) and \(c\) are functions of \(x, y,\) and \(u\) and \(u = u(x, y)\) is given on an initial curve \(\Gamma.\) Derive the equations satisfied by the characteristic curves.

(b) Suppose
\[ u_x + 2xu_y = x, \quad y > 0, \quad -\infty < x < \infty \]
\[ u(x, 0) = 4 \quad -\infty < x < \infty \]
Calculate the value of \(y\) so that \(Q(3, y)\) is on the characteristic curve through \(P(2, 0)\).

(c) Compute the exact value \(u_Q,\) where \(Q\) is the point found in (b).

(d) Use the method of numerical characteristic to calculate first approximations to the value of \(y_Q\) and \(u_Q.\)

2. Consider the problem
\[ u_t = 3u_{xx} \quad 0 < x < 1, t > 0 \]
\[ u(x, 0) = f(x) \quad 0 \leq x \leq 1, f(x) \text{ given} \]
\[ u(0, t) = u(1, t) = 0 \quad t > 0. \]
Suppose we approximate the PDE by the finite difference equation
\[ \frac{u_{i,j+1} - u_{i,j}}{k} = 3 \left[ \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \right] \]
where \(u_{i,j} = u(ih, jk), h = \Delta x,\) and \(k = \Delta t.\)

(a) Show that the finite difference equation is consistent with \(u_t = 3u_{xx}.\)

(b) Let \(h = 1/5, r = k/h^2.\) Find the \(4 \times 4\) matrix such that
\[ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_{j+1} = A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_j, \quad j = 0, 1, 2, \ldots \]

(c) For what values of \(r = k/h^2\) is the scheme stable? Explain.
3. Consider the PDE
\[ u_{xx} + u_{yy} = 0, \quad 0 < x < 1, 0 < y < 1 \]
\[ u(x, 0) = x^2, \quad u(x, 1) = x^2 - 1 \quad 0 \leq x \leq 1 \]
\[ u(0, y) = -y^2, \quad u(1, y) = 1 - y^2 \quad 0 \leq y \leq 1 \]

(a) Show that \( u(x, y) = x^2 - y^2 \) is the exact solution to this problem.

(b) Find the maximum and minimum values of \( u(x, y) \) and say at what point they occur.

(c) Using the standard 5-point difference scheme for approximating the PDE write out the equations you get for \( \Delta x = \Delta y = 1/3 \). Simplify them.

(d) For arbitrary \( \Delta x \) and \( \Delta y \) explain how you know the equations in part (c) have a unique solution.