Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers. 
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number $z$.
$\text{Im}(z)$ denotes the imaginary part of the complex number $z$.
$\overline{z}$ denotes the complex conjugate of the complex number $z$.
$|z|$ denotes the absolute value of the complex number $z$.
$\text{Log } z$ denotes the principal branch of $\log z$.
$\text{Arg } z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center $z$ and radius $r$.
A domain is an open connected subset of $\mathbb{C}$.

**MISCELLANEOUS FACTS**

\[ 2 \sin a \sin b = \cos(a - b) - \cos(a + b) \]
\[ 2 \cos a \cos b = \cos(a - b) + \cos(a + b) \]
\[ 2 \sin a \cos b = \sin(a + b) + \sin(a - b) \]
\[ 2 \cos a \sin b = \sin(a + b) - \sin(a - b) \]
\[ \sin(a + b) = \sin a \cos b + \cos a \sin b \]
\[ \cos(a + b) = \cos a \cos b - \sin a \sin b \]
\[ \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \]
\[ \sin^2 a = \frac{1}{2} - \frac{1}{2} \cos(2a) \]
\[ \cos^2 a = \frac{1}{2} + \frac{1}{2} \cos(2a) \]
Winter 2002 # 1. Describe and sketch each of the following sets

a. \[ A = \{ z \in \mathbb{C} : \text{Re}\left(\frac{1}{z}\right) > \frac{1}{2} \} \]

b. \[ B = \{ z \in \mathbb{C} : \left|\frac{z-2}{z+1}\right| > 2 \} \]

Winter 2002 # 2. For \( z \) in \( \mathbb{C} \), let \( z = x + iy \) with \( x \) and \( y \) real. For each of the following real valued functions \( u(x, y) \), determine whether there is a real valued function \( v(x, y) \) such that the function \( f(z) = u(x, y) + iv(x, y) \) is analytic and \( f(0) = i \). If there is such a function \( v \), find one and explain how you know that \( f \) is analytic. If there is not, explain how you know that there is not.

a. \( u(x, y) = (x + 1)y \)
b. \( u(x, y) = (x + y)y \)

Winter 2002 # 3. Let \( f(z) = \frac{z^2 - 1}{\sin \pi z} \)

a. Find all singularities of \( f \) in \( \mathbb{C} \) and classify each as a pole (specifying the order), essential, removable, or other.

b. Explain why \( f(z) \) has a series expansion of the form \( \sum_{k=-\infty}^{\infty} c_k z^k \) valid for \( z \) near 0. Which, if any, of the coefficients \( c_k \) for \( k < 0 \) are not equal to 0?

c. Find \( c_{-1} \), \( c_0 \), and \( c_1 \)
d. What is the region of validity for the expansion discussed in part b?

e. Find \( \int_\gamma f(z) \, dz \) where \( \gamma \) is the circle of radius 1 centered at the origin and travelled once counterclockwise.

Winter 2002 # 4. Evaluate the following integrals using complex variable methods. Show any curves and explain any estimates needed to justify your method.

a. \( \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} \); b. \( \int_{-\infty}^{+\infty} \frac{dx}{x^2 - 4x + 5} \); c. \( \int_0^\infty \frac{\sqrt{x}}{1 + x^2} \, dx \)

Winter 2002 # 5. Let \( p \) be a polynomial with \( p(0) = 0 \).

a. Evaluate \( \int_{-\pi}^{\pi} (1 - p(e^{i\theta})) \, d\theta \)

b. Show that there is at least one real number \( \theta \) with \( |1 - p(e^{i\theta})| \geq 1 \)
Winter 2002 # 6. Let $m$ and $n$ be integers with $m > n > 0$. Let $q(z)$ and $p(z)$ be polynomials of degree $m$ and $n$

\[ p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad \text{and} \quad q(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m \]

Let $\gamma_R$ be the circle of radius $R$ centered at 0 and travelled once counterclockwise. Show that

\[ \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{p(z)}{q(z)} \, dz = \begin{cases} 
\frac{a_0}{b_0}, & \text{if } m = n + 1 \\
0, & \text{if } m - n > 1 
\end{cases} \]

Winter 2002 # 7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$ and that there is a polynomial $p$ of degree $n$ and a point $z_0$ such that $|f(z)| \leq |p(z)|$ for all $z$ with $|z| \geq |z_0|$.

Prove that $f$ must be a polynomial of degree no more than $n$.

End of Exam